

# Stability of Cosmic Strings

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A thesis submitted for the degree of

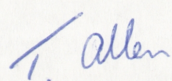
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Chapter 1 contains summaries of the established work in the field to set the background for the thesis. The remainder of the thesis, except where the work of others is explicitly cited, is either my own work or the result of collaboration between myself and Dr L.J.Tassie. The results reported in section 1.4 and chapters 2,3,6 and 7 are my work, although I frequently discussed the work with Dr Tassie, who gave me advice and helped me form ideas. The numerical work reported in chapters 4 and 5 is mine, while the algebraic work leading up to it was the result of our collaboration.



Timothy John Allen

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# Publications

Some of the work in this thesis has been previously reported in the following publications:

**T.J.Allen**, “Stable superconducting cosmic strings”, *Phys. Lett. B* **231** (1989) 429.

**T.J.Allen**, “Charge separation on colliding superconducting cosmic strings”, *Phys. Lett. B* **250** (1990) 29.

**T.J.Allen and L.J.Tassie**, “Spin-one vacuum instability around a charged superconducting cosmic string”, submitted to *J. Phys. G*.



# Abstract

Cosmic string theories have been used to attempt to explain features of cosmology that are not accounted for in the standard big-bang model. Previous work has established the possibility of superconducting cosmic string loops being stabilized by their currents. It is shown here that charged loops may also be stabilized. This is important because there is an explicit limiting mechanism that applies to current but not to charge. Charge can thus be more important than current in stabilizing loops, as long as a mechanism for separating significant charge exists.

Such a mechanism, collision of strings in relativistic motion, is explored and found to separate charge significant compared with the currents on the strings. Stable charged loops may thus be a dominant feature of a cosmology that involves superconducting cosmic strings.

Because charged strings may be sources of strong electromagnetic fields, the possibility of vacuum instability in the neighbourhood of a charged string is examined. It is found that a charged condensate of pions could form around such a string, acting to shield the charge of the string from the rest of the universe. Electrons are also found to have the potential to cause a vacuum instability, though it is not clear how important this would be compared to the pion condensate. The possibility of W-boson condensation in the field of a charged string is also considered.

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# Chapter 1

## Introduction

### 1.1 Cosmology and particle physics

At first glance it might be expected that two such seemingly disparate fields as particle physics and cosmology would have little to do with each other. However, to a greater and greater extent, the two are interacting, particularly in the study of the early universe, where energy regimes far in excess of those accessible in particle accelerators prevailed. Cosmology has become a laboratory for particle physicists and, conversely, particle physics is contributing explanations of phenomena for which conventional cosmology is unable to account.

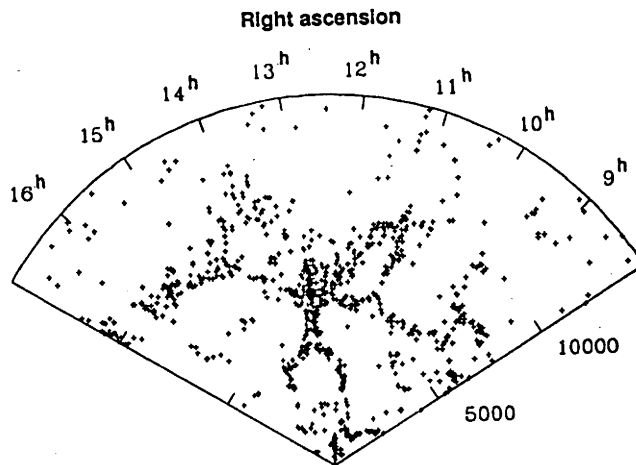


Figure 1.1: Large-scale distribution of galaxies.

## 1.2 Cosmic strings

A persistent question in cosmology is the structure problem. Conventional cosmology predicts that as the universe is observed on larger and larger scales, it should appear more and more homogeneous. This is not, however, what is observed; the universe apparently has structure on all scales. Large-scale galaxy surveys [1,2,3] show that there are sheet and filament-like regions where there is a high density of galaxies and voids where there is a very low galaxy density. Figure 1.1, taken from reference [4], illustrates this.

Cosmic string theory attempts to resolve this problem by invoking the topological defects that arise from spontaneously broken gauge symmetries in quantum field theory [5]. From the point of view of a classical cosmologist, a cosmic string is simply a massive, one-dimensional manifold. Because they can have no ends, cosmic strings occur either as loops or as long strings that stretch across the entire universe.

The gravitational properties of cosmic strings are of interest in cosmology because their large mass-density, of the order of  $10^{22}$  g/cm [6], would produce large gravitational effects. These gravitational fields would cause matter to accrete in the neighbourhood of cosmic strings, providing the inhomogeneities that produce the structure apparent on large scales.

The gravitational field of a cosmic string is not a simple Newtonian attraction. The attraction is, in fact, quite weak, and depends on the curvature of the string, since a perfectly straight, stationary string has no attractive gravitational field at all [7,8,9,10,11,12]. However, the space-time around a cosmic string is far from being unaltered by the presence of the string. The surrounding space-time has no curvature, but has an angle deficit,  $\alpha = 8\pi\mu$ , where  $\mu$  is the string tension, or mass per unit length, in natural units. What this means is that a circle drawn with the string



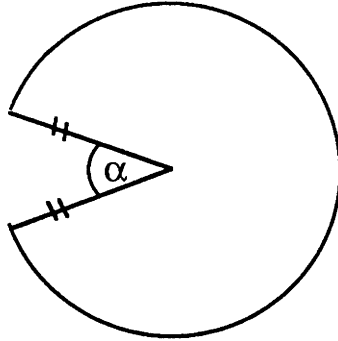


Figure 1.2: The space-time around a straight cosmic string has a conical singularity, with an angle deficit  $\alpha$ .

passing through its centre will encompass an angle of not  $2\pi$ , but  $2\pi - \alpha$ .

The truly significant gravitational effects come from a cosmic string in motion. Since the string tension equals the mass per unit length, cosmic strings continually undergo large accelerations due to their own tension, and routinely travel with highly relativistic velocities. As a cosmic string passes through a region, the deformation of space due to its gravitational field causes matter to accrete in its wake, creating sheet-like structures [13]. Accretion on all scales can be seeded by cosmic string motion in this way, so cosmic strings may offer an explanation of the structure of the universe, from structures like the ‘Great Wall’ [4] down to galaxy formation.

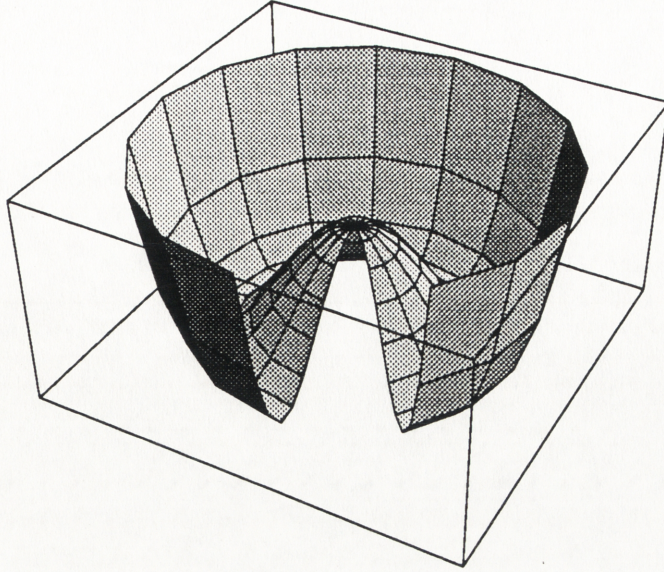


Figure 1.3: ‘Wine-bottle’ potential of  $U(1)$  scalar field at zero temperature.

To a particle physicist, a cosmic string is a vortex-like topological defect [5] produced during spontaneous symmetry breaking in a gauge theory. Any broken symmetry that forms a non-simply-connected groundstate may produce cosmic strings, but it is easiest to demonstrate the principles by use of a  $U(1)$  gauge theory as an example. Including the effects of temperature, the effective potential for a complex scalar field,  $\phi$ , with  $U(1)$  gauge symmetry is [14]

$$V_T(\phi) = \frac{1}{4}\lambda(\phi^*\phi - \sigma^2)^2 + kT^2\phi^*\phi. \quad (1.1)$$

A projection of this potential is shown in figure 1.4. For high temper-

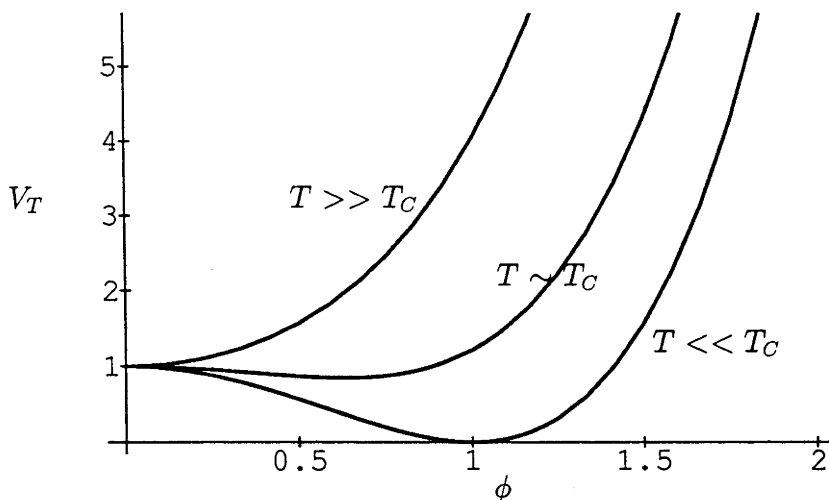


Figure 1.4: Projected effective potential for a  $U(1)$  complex scalar field at varying temperatures.

atures, the parabolic term,  $kT^2\phi^*\phi$ , dominates, and the groundstate is the point,  $\phi = 0$ . As the temperature falls, however, the first term dominates more and more until a circular manifold of possible groundstates forms, approaching  $\phi = \sigma e^{i\theta}$  as  $T$  tends towards 0. To see how cosmic strings arise from such a groundstate, consider a circular loop,  $C_r$ , in space, with radius  $r$ . The minimum energy-density field values occur within the manifold,

$$M = \{\sigma e^{i\alpha}, 0 \leq \alpha \leq 2\pi\}. \quad (1.2)$$

Assume that  $\phi$  takes on one of these minimal energy-density values at each point,  $x = re^{i\theta}$ , of the loop, and that  $\alpha$  increases from 0 to  $2\pi$  as  $x$

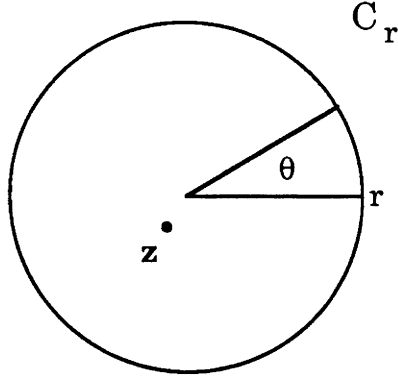


Figure 1.5: With a nontrivial winding number about the circle  $C_r$ , the field  $\phi$  must be zero at some point,  $\mathbf{z}$ , on the disk bounded by  $C_r$ .

winds around the loop, for example

$$\phi(r, \theta) = re^{i\theta}. \quad (1.3)$$

Another way of saying this is that the field configuration has a winding number of 1 around the circle  $C_r$ . Now, if  $\phi(\mathbf{x})$  is continuous, there must be some point  $\mathbf{z}$  on the disk bounded by  $C_r$  where  $\phi(\mathbf{z}) = 0$ . As the loop  $C_r$  is continuously deformed, it must be possible to shrink it to a point where  $\phi$  takes on all phase values in  $M$ . This must imply that the phase of  $\phi$  is unimportant at that point, which is only true for  $\phi = 0$ . It may be shown that a line of defect points, rather than just one point, is formed, by considering deformations of the disk bounded by  $C_r$  in the direction



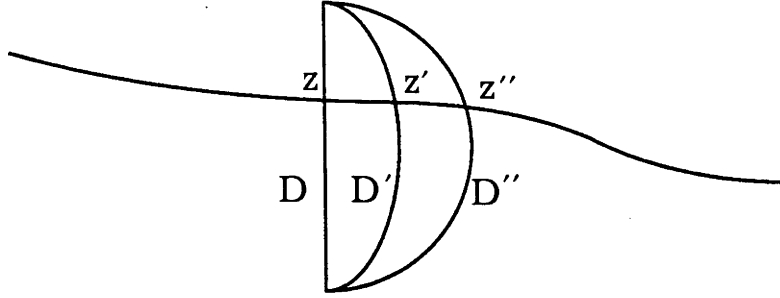


Figure 1.6: The disk  $D$  contains a point  $z$  at which  $\phi = 0$ . Continuously deforming  $D$  must preserve this property, so the disks  $D'$  and  $D''$  must also have points  $z'$  and  $z''$  where  $\phi = 0$ .

perpendicular to it. The same argument may be applied to each of these deformed disks, and each deformed disk must have a point where  $\phi = 0$ . Thus, there must be a line of points where  $\phi = 0$ , and because the potential has a large value there, this line must correspond to a large linear mass density. Note that the value of the field,  $\phi = 0$ , at the core of the string, is the same as the vacuum field value before the phase transition that caused cosmic strings to form. The cosmic string is, in effect, then, a piece of primeval vacuum trapped by topology and unable to reach the currently prevalent vacuum state.

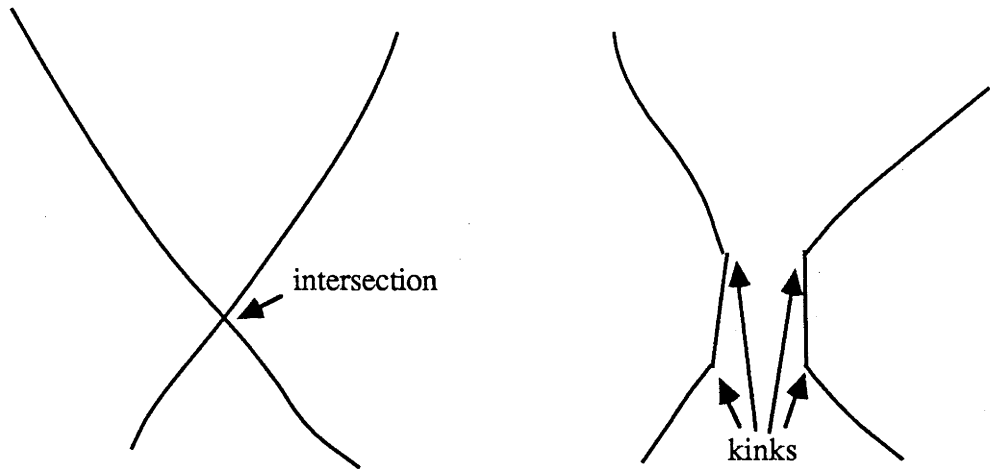


Figure 1.7: Strings intercommute on collision, forming new strings.

The initial phase transition would, by the Kibble mechanism [15], cause a network of cosmic strings throughout the entire universe to come into being. At first, this network would consist predominantly of long strings, each stretching across the universe. These long strings would collide with each other and intercommute [16], forming loops of string. The new strings formed after a collision would each have a pair of kinks, where the shape of the string is not smooth, because in general the strings will collide at some angle. The kinks progress around the string at the speed of light, and effectively transmit to the rest of the string the information that a collision has occurred.

Early simulations [17] of a cosmic string network suggested that the

‘scaling solution’ obtained from simple analytical models [18] would arise, whereby the network is self-similar on all scales. More recent simulations [19,20,21,22], which model string behaviour on smaller scales, tend to suggest that kinks are more important and may cause more small loops to be produced than the earlier work predicted. Also, loops would tend to travel with greater velocities, so any cosmological scenario that relies on loops staying in more-or-less the same place for any significant period of time is untenable.

Another important issue in cosmic string theories is the formation of cusps. The simple dynamics of a free cosmic string allow the possibility of cusps, which are points where part of the string instantaneously (or in some cases, persistently) travels with the speed of light. Such features, if they occurred, would dominate string evolution because they would radiate huge amounts of gravitational (and electromagnetic, in the case of superconducting cosmic strings) radiation. Simple free-string trajectories with no kinks generically form cusps [23]. However, since kinks are always produced in the collisions which form loops, cusps are not required to occur. Radiation reaction from cusps might also tend to smooth them out, though this has not been established definitively. It is also possible that radiation

may smooth out kinks.

### 1.3 Superconducting cosmic strings

In addition to their gravitational properties, cosmic strings may, under some circumstances, superconduct, and thus exhibit interesting electromagnetic properties. Two mechanisms permit superconductivity on cosmic strings, one fermionic and one bosonic [24]. Fermionic superconductivity may occur when fermionic particles gain their mass by coupling to the gauge field responsible for the existence of the cosmic string. Because the core of the string represents, in effect, a piece of space in the vacuum state prevalent before the string-creating phase transition, the fermions would have no mass there. A fermion in the core of the string would be confined to the neighbourhood of the string by the mass difference, as it would need to acquire energy equal to its true-vacuum mass to leave. Such a fermion, being massless, would travel along the string at the speed of light and, if charged, would constitute a supercurrent. The current possible with this mechanism is limited by the Pauli exclusion principle, scattering of the fermions from each other, and tunnelling. For particles as massive as

electrons, Witten [24] estimates that the maximum current would be of the order of 20 Amps. Superheavy fermions, such as supersymmetry partners, might permit currents as high as  $10^{20}$  Amps.

Bosonic superconductivity may occur when the gauge field responsible for the existence of the strings is coupled to another field in a way that is affected by its broken symmetry. For example, consider a  $U(1) \times U(1)$  gauge theory in which one  $U(1)$  is broken, and responsible for the string, and the other is not [24]. If the broken  $U(1)$  has a scalar field  $\phi$ , we know from the previous section that  $\phi = 0$  in the core of the string but  $\phi \neq 0$  away from the string. Since the other  $U(1)$  is unbroken, its associated scalar field,  $\sigma$ , say, is zero in ordinary vacuum. However, in some models, the interaction between the fields may require that  $\sigma \neq 0$  in the core of the string. Thus, the minimum energy state could have  $\sigma = \sigma_0 e^{i\theta}$  in the string core.

The field  $\sigma$  may develop excitations. Variations in  $\sigma$  from point to point along the string would correspond to a current flow around the string. If  $\sigma$  has a winding number with respect to a circuit of the string loop, the current would have to be persistent because the winding number is a topologically preserved quantity. It should be noted that this winding number is with respect to a closed loop *along* the string, not a loop in space encompassing

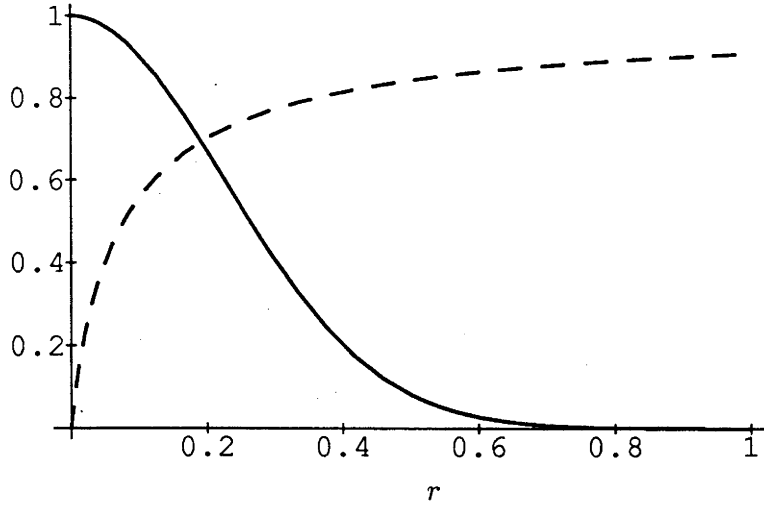


Figure 1.8: Profiles of the fields of a bosonic superconducting cosmic string,  $\phi$ , the field responsible for the string, dashed line, and  $\sigma$ , the field carrying the current, full line.

a part of the string as was discussed in the previous section to explain the existence of ordinary cosmic strings. The maximum current, limited by tunnelling processes, is estimated by Witten [24] to be of the order of  $10^{20}$  Amps.

This persistent current is not necessarily an electromagnetic current, though there are certainly models where it is. A non-electromagnetic current would have interesting effects, though most attention has focussed on electromagnetic currents. The terminology used in this thesis reflects this common focus, although some of the results derived herein are equally valid

for non-electromagnetic currents, as they depend on the kinetic, rather than electromagnetic, properties of the current. Fermionic superconductivity is likewise not confined to electromagnetism, because the trapped fermions may possess some other sort of charge than electromagnetic charge.

Superconducting cosmic strings have been used to construct a structure formation scenario quite different to the usual gravitational accretion model. It has been suggested by Ostriker, Thompson and Witten (OTW) [25] that vibrating superconducting cosmic strings would emit large amounts of radiation. This radiation would exert pressure on the surrounding matter, heating it and repelling it away from the string and, in effect, blowing 'bubbles'. The heated gas cools as it recedes from the string and gravitational processes take over, forming stars and galaxies. Regions of low density thus correspond to the vicinity of a superconducting cosmic string, and regions of high mass density correspond to the boundaries of the voids created by the radiation pressure.

Earlier observations of large-scale structure [1] gave some support to this picture, as it appeared that the universe had a foamy structure that corresponded well with the bubbles that radiation pressure from superconducting cosmic strings would create. Further observations, however, have

shown that the picture is not so simple, and that sheet-like structures are common [4]. It has also been realised that cosmic strings would not be stationary, but would move with relativistic velocities. It is thus not clear what picture would really evolve from superconducting cosmic strings radiating large amounts of electromagnetic power. Structure formed would be continually rearranged by the passage of other strings, and a simple structure of voids with galaxies forming on the edge of the voids would not necessarily arise. The observational status of the superconducting cosmic string theory is hence unclear.

Since OTW's work, a number of authors have investigated the properties of superconducting cosmic strings [26,27,28,29,30,31] and have shown that they are much more complicated, and interesting, objects than was apparently envisaged by OTW. The present work continues and extends this investigation, and reinforces the evident complexity of superconducting cosmic strings.

Electromagnetic self-interactions of superconducting cosmic strings have been modelled numerically [32,33], indicating the possibility that such systems may be chaotic. The evidence is not conclusive, however, as the work implicitly assumed the presence of cusps on the strings. It is not clear that



cusps will actually occur on superconducting cosmic strings, because kinks would be common and because, as chapter 2 will show, the presence of charge-carriers on the string tends to reduce curvature. What is clear is that electromagnetic interactions complicate matters and ultimately must be taken into account. The problems studied in the present work, however, turn out to have more than adequate complexity without electromagnetic self-interaction, and it was not possible to incorporate an adequate treatment of electromagnetic interactions.

## 1.4 Gravitational Radiation

The dominant decay mechanism for cosmic strings is gravitational radiation. The rate of gravitational radiation determines the lifetime of the loop, the details of the string dynamics and could contribute to a gravitational radiation background observable in pulsar timings [34].

It has been shown that for a cosmic string with string tension  $\mu$ , the power emitted in gravitational radiation is

$$\frac{dE}{dt} = \gamma G \mu^2 \tag{1.4}$$

where  $\gamma$  is a constant that depends of the details of the string trajectory and

is typically of the order of 100 [35,36]. The total energy of a cosmic string of length  $L$  is  $\mu L$ , so the lifetime of a string is proportional to  $L$ . Since  $G\mu \sim 10^{-6}$ , a string lifetime is typically of the order of  $10^4 L$ . This implies that the infinite strings and the largest string loops should still exist, while smaller ones are likely to have evaporated by now.

Two approaches to studying gravitational radiation were examined by the author and, although the results were somewhat negative, they are reported here. The objective was to extend previous results [35], which, although providing estimates of the power emitted by gravitational radiation, provided no estimate of the angular momentum or momentum radiated.

Thorne [37] gives a formula for the energy and angular momentum radiated gravitationally by a system using the quadrupole approximation. As this is a small-velocity approximation, it wouldn't be expected to give accurate results for a relativistic system such as a cosmic string, but it might be expected to give reasonable order-of-magnitude results. The energy radiated is

$$\frac{dE}{dt} = \frac{1}{5} \sum_{jk} \left\langle \left( \frac{\partial^3 J_{jk}}{\partial t^3} \right)^2 \right\rangle \quad (1.5)$$

and the angular momentum loss is

$$\frac{dJ_i}{dt} = \frac{2}{5} \sum_{jkl} \epsilon_{ijk} \left\langle \frac{\partial^2 \mathcal{J}_{jl}}{\partial t^2} \frac{\partial^3 \mathcal{J}_{lk}}{\partial t^3} \right\rangle \quad (1.6)$$

with the quadrupole moments defined by

$$\mathcal{J}_{jk} = \int \rho(t) \left( x_j x_k - \frac{1}{3} r^2 \delta_{jk} \right) d^3 \mathbf{x}. \quad (1.7)$$

The Kronecker delta,  $\delta_{jk}$ , and alternating tensor,  $\epsilon_{jkl}$ , have their usual definitions and  $\rho$  is the mass density at various points within the system. The angle brackets,  $\langle \dots \rangle$ , denote a time average over one period of the system's motion.

The radiation from the Burden trajectories [35] was calculated this way.

The general solution to the free cosmic string equations of motion is

$$\mathbf{r}(\tau, \sigma) = \frac{L}{4\pi} [\mathbf{a}(\xi) + \mathbf{b}(\eta)] \quad (1.8)$$

where

$$\xi = \frac{2\pi}{L}(\sigma - \tau)$$

$$\eta = \frac{2\pi}{L}(\sigma + \tau)$$

$$\mathbf{a}(s + 2\pi) = \mathbf{a}(s)$$

$$\mathbf{b}(s + 2\pi) = \mathbf{b}(s)$$

and

$$\mathbf{a}'^2 = \mathbf{b}'^2 = 1.$$

The solutions in reference [35] now specify that

$$\mathbf{a}(\xi) = (M^{-1} \sin M\xi, 0, M^{-1} \cos M\xi) \quad (1.9)$$

and

$$\mathbf{b}(\eta) = (N^{-1} \cos \psi \sin N\eta, N^{-1} \sin \psi \sin N\eta, N^{-1} \cos N\eta), \quad (1.10)$$

for integers  $M$  and  $N$  and an angle  $\psi$ . It is convenient to define

$$a = \frac{2\pi M}{L}$$

and

$$b = \frac{2\pi N}{L}.$$

Then the moments are

$$\begin{aligned} \mathcal{J}_{11} &= \frac{\mu L}{12} \left[ \frac{1}{2a^2} + \frac{3 \cos^2 \psi - 2}{2b^2} + \right. \\ &\quad \left. \frac{\cos 2a\tau}{ab} \{ \delta_{ab}(2 \cos \psi - 1) - \delta_{a,-b}(2 \cos \psi + 1) \} \right] \\ \mathcal{J}_{12} &= \frac{\mu L}{8} \left[ \frac{\sin \psi}{ab} \cos 2a\tau (\delta_{ab} - \delta_{a,-b}) + \frac{\sin \psi \cos \psi}{b^2} \right] \\ \mathcal{J}_{13} &= \frac{\mu L}{8ab} \sin 2a\tau [\delta_{ab}(\cos \psi - 1) - \delta_{a,-b}(\cos \psi + 1)] \\ \mathcal{J}_{21} &= \mathcal{J}_{12} \end{aligned}$$

$$\begin{aligned}
\mathcal{J}_{22} &= \frac{\mu L}{12} \left[ -\frac{1}{a^2} \frac{3}{2b^2} \sin^2 \psi - \frac{\cos 2a\tau}{ab} \{ \delta_{ab}(1 + \cos \psi) + \delta_{a,-b}(1 - \cos \psi) \} \right] \\
\mathcal{J}_{23} &= \frac{\mu L}{8ab} \sin \psi \sin 2a\tau (\delta_{ab} - \delta_{a,-b}) \\
\mathcal{J}_{31} &= \mathcal{J}_{13} \\
\mathcal{J}_{32} &= \mathcal{J}_{23} \\
\mathcal{J}_{33} &= \frac{\mu L}{12} \left[ \frac{1}{2a^2} + \frac{1}{2b^2} + \frac{\cos 2a\tau}{ab} \{ \delta_{ab}(2 - \cos \psi) + \delta_{a,-b}(2 + \cos \psi) \} \right].
\end{aligned} \tag{1.11}$$

Unless  $a = \pm b$ , the quadrupole moments are all independent of time, and there is no radiation. For  $a = b$ ,

$$\frac{dE}{dt} = \frac{\mu^2 a^2 L^2}{15} (\cos^2 \psi - 10 \cos \psi + 13) \tag{1.12}$$

and

$$\frac{d\mathbf{J}}{dt} = \frac{\mu^2 a L^2}{5} \left( \frac{1}{2} (\sin 2\psi - 6 \sin \psi), -\cos^2 \psi + 4 \cos \psi - 3, 0 \right). \tag{1.13}$$

If  $a = -b$ ,

$$\frac{dE}{dt} = \frac{\mu^2 a^2 L^2}{15} (\cos^2 \psi + 10 \cos \psi + 13) \tag{1.14}$$

and

$$\frac{d\mathbf{J}}{dt} = \frac{\mu^2 a L^2}{5} \left( \frac{1}{2} (\sin 2\psi + 6 \sin \psi), -\cos^2 \psi - 4 \cos \psi - 3, 0 \right). \tag{1.15}$$

Other work [35,38] has shown that for trajectories corresponding to  $a \neq \pm b$ , there is indeed a significant amount of energy radiated away gravitationally, and thus these quadrupole results are inadequate. While it would be expected that the quadrupole approximation would not give accurate results, it comes as something of a surprise that the approximation does not even get the orders of magnitude correct.

The second approach to gravitational radiation taken was a numerical method using a post-Minkowski approximation [39]. The essence of the approximation is to regard the space-time metric as a first-order perturbation from a Minkowski metric. As this does not require slow motion for validity, it may be expected to perform considerably better than quadrupole or post-Newtonian approximations. Thus

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (1.16)$$

Expanding Einstein's equation to first order in  $h_{\mu\nu}$  gives

$$h_{\mu\nu}(t, \mathbf{x}) = 4 \int d^3\mathbf{x}' \frac{T_{\mu\nu}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (1.17)$$

where  $t' = t - |\mathbf{x} - \mathbf{x}'|$ . The  $i$ th derivative of the metric perturbation would thus be

$$h_{\mu\nu,i} = 4 \int d^3\mathbf{x}' \left[ \frac{(x_i - x'_i)}{|\mathbf{x} - \mathbf{x}'|^2} T_{\mu\nu,0}(t', \mathbf{x}') - \frac{(x_i - x'_i)}{|\mathbf{x} - \mathbf{x}'|^3} T_{\mu\nu}(t', \mathbf{x}') \right]. \quad (1.18)$$

At distances,  $r = |\mathbf{x}|$ , large compared to the maximum spatial extent of the source,  $|\mathbf{x} - \mathbf{x}'| \approx r$  and

$$h_{\mu\nu} \approx \frac{4}{r^2} \int d^3\mathbf{x}' (x_i - x'_i) \left\{ T_{\mu\nu,0}(t', \mathbf{x}') - \frac{T_{\mu\nu}(t', \mathbf{x}')}{r} \right\}. \quad (1.19)$$

The energy-momentum tensor of a cosmic string with trajectory  $\mathbf{r}(t, \sigma)$  is

$$\begin{bmatrix} T_{00} & T_i \\ T_j & T_{ij} \end{bmatrix} = \mu \int d\sigma \delta^3(\mathbf{x} - \mathbf{r}(t, \sigma)) \begin{bmatrix} 1 & \dot{r}_i \\ \dot{r}_j & \dot{r}_i \dot{r}_j - r'_i r'_j \end{bmatrix}. \quad (1.20)$$

The energy-momentum tensor for the gravitational radiation may be calculated from the transverse-traceless part of the metric perturbation [40,41] via

$$T_{\mu\nu} = \frac{1}{32\pi} \langle h_{jk,\mu}^{TT} h_{jk,\nu}^{TT} \rangle. \quad (1.21)$$

Defining the projection operator,

$$P_{jk} = \delta_{jk} - n_j n_k$$

with  $n_j = \frac{x^j}{r}$ , one can obtain the transverse-traceless metric perturbation from [39]

$$h_{jk}^{TT} = P_j^a P_k^b h_{ab} - \frac{1}{2} P_{jk} P_{ab} h^{ab}. \quad (1.22)$$

A computer program to calculate the metric perturbation and energy-momentum tensor due to gravitational radiation was written and, although

the preliminary results were encouraging, the algorithm was sufficiently computationally intensive that it proved infeasible to continue calculations using the computing facilities available to the author at that time. Some of the preliminary results are shown, however, in figure 1.9, which depicts the metric component  $h_{23}$  at a particular point in space as a function of time, using one of the trajectories from reference [35] as the source. It can clearly be seen that the dominant frequency of the gravitational wave is  $\frac{4\pi MN}{L}$ , in accordance with previous results [35].

More recently, momentum and angular momentum of gravitational radiation from cosmic strings has been calculated by Durrer [38] using an analytic and computational technique that improves significantly on that attempted here.



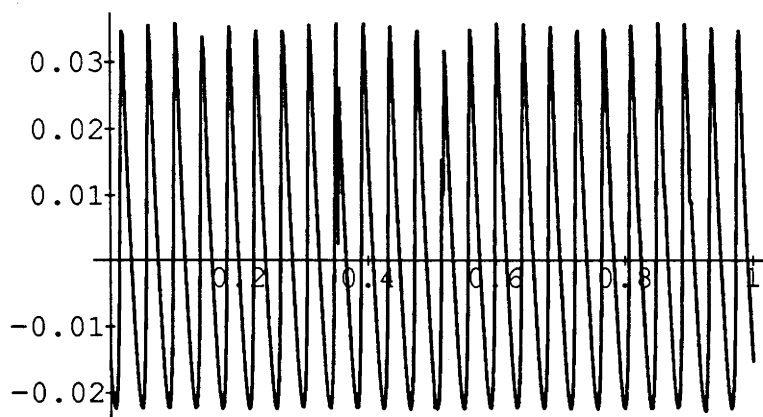


Figure 1.9: Metric perturbation,  $h_{23}$ , as a function of time, at  $\mathbf{x} = (10000, 10000, 10000)$  in units of  $L = 1$ . The source was the Burden trajectory with  $\psi = \frac{\pi}{6}$ ,  $M = 3$ ,  $N = 4$ .

## Chapter 2

# Stable Superconducting

# Cosmic Strings

### 2.1 Introduction

It has been shown by several authors [31,28,26,29] that it is possible to have a stable or static superconducting cosmic string. Such objects would have significant lifetimes and may present problems for cosmological scenarios. For this reason it has been suggested [29] that theories where such stable loops may arise be eliminated from contention. In particular, if the maximum current permitted on the string is not sufficient to stabilize it

at a significant radius, the theory may be pronounced safe. However, as this chapter aims to show, the stable trajectories in the literature may be generalized and stability must be seen as a more ubiquitous problem for superconducting cosmic string theories. It is not yet clear how serious a problem this is, but it casts doubt on the Ostriker, Thompson and Witten [25] scenario of structure formation.

## 2.2 Current and Charge

A superconducting cosmic string may acquire its current via a number of mechanisms, the simplest of which is motion of the string through primordial magnetic fields. The induced electric field would establish current flow along the string. A more detailed treatment of this and other mechanisms is given by Mijić [42]. With current on the string, relativistic effects will produce local charge distributions. When such strings in relativistic motion collide and intercommute, the new strings formed will in general possess a nett charge.

The current on the string will persist because of the superconductivity mechanisms demonstrated by Witten [24] and any nett charge should

persist because the charge-carriers are trapped on the string. However, to assert this with confidence one must consider whether the surrounding plasma will interact significantly with the confined charges.

If, as is likely to be the case, the string is moving quickly compared with the speed of sound in the plasma, then according to Chudnovsky, Field, Spergel and Vilenkin [43] a shock wave is established which denudes the immediate vicinity of the string of plasma. This would tend to prevent interaction between the plasma and charges on the string. For subsonic motion, there may be a significant plasma density around the string so interactions with the plasma might occur.

However, collisions between the string and plasma particles would be hindered by the very small thickness of the string, tiny even compared with the effective spatial extent of a proton. For this reason, protons could not provide an effective collisional mechanism for neutralizing strings. Even if electrons were able to collide effectively, it would still only be possible for positively charged strings to be neutralized in this way.

The charge-carriers on the string may have fractional charge, which would make it difficult for ordinary matter to neutralize them. For example, if the charge-carriers have  $+2/3e$  charge then it would be necessary for

two electrons to collide with three string-confined charges. Such five-body reactions are unlikely to occur at significant rates.

It does not seem likely, then, that interaction with the surrounding plasma will cause the string to lose any net charge at a significant rate.

## 2.3 Notation

The notation and equations of motion used in this chapter are based on the work of Spergel, Piran and Goodman (SPG) [28]. The string world-sheet is parametrized by two variables,  $\xi^0 = \tau$ ,  $\xi^1 = \sigma$ . It is usually assumed that  $\tau$  is time-like and  $\sigma$  is space-like. Points on the world-sheet are located by the four-vector  $Z^\mu(\tau, \sigma)$ ,  $\mu = 0, \dots, 3$ . Although it is possible in some cases, one cannot in general identify the time,  $Z^0$ , with  $\tau$ . The string tension is denoted  $T$ , rather than  $\mu$  as used elsewhere and by other authors, to avoid any possible confusion with the four-vector index.

The metric induced on the world-sheet surface is defined by

$$h_{ij} = \partial_i Z^\mu \partial_j Z_\mu, \quad i, j = 0, 1$$

The determinant of this metric is  $h = h_{00} h_{11} - h_{01}^2$ .

Electric current on the string is described by a two-vector,  $J^i$ ,  $i = 0, 1$ .

In a somewhat imprecise sense one can call  $J^0$  the charge and  $J^1$  the current on the string. The two-current is induced by a scalar field,  $\phi$ , by

$$J^0 = q \frac{\partial_1 \phi}{\sqrt{-h}}, \quad J^1 = -q \frac{\partial_0 \phi}{\sqrt{-h}}$$

where  $q$  is the charge of the fermionic or bosonic charge carriers. This form of the field inducing the current more naturally reflects the presence of fermionic charge carriers, but for bosonic charge carriers the current may be expressed in the same mathematical form and the equations of motion used here are equivalent to those derived by Copeland, Hindmarsh, Haws and Turok (CHHT) [29] for bosonic superconductivity.

The gauge used is the conformal gauge, which requires that

$$h_{00} + h_{11} = 0$$

and

$$h_{01} = 0. \tag{2.1}$$

For the purposes of this work, the electromagnetic self-interaction of the string will be neglected. A full treatment of the self-interaction would need to take into account the effect of the surrounding plasma, because at velocities small compared with the speed of sound in the plasma Debye screening

would electromagnetically shield the string from itself, depending on how large the string is compared to the Debye length of the plasma, while at large velocities the effect of shock waves would have to be considered. Neglecting self-interaction should be a good approximation for slowly moving strings. It remains to be seen whether it is adequate for faster strings.

Without electromagnetic interactions, the equations of motion in the conformal gauge are [28]

$$\begin{aligned} \partial_0 \left[ \left( T + \frac{\partial_0 \phi^2 + \partial_1 \phi^2}{2h_{11}} \right) \partial_0 Z^\nu - \frac{\partial_0 \phi \partial_1 \phi}{h_{11}} \partial_1 Z^\nu \right] \\ - \partial_1 \left[ \left( T - \frac{\partial_0 \phi^2 + \partial_1 \phi^2}{2h_{11}} \right) \partial_1 Z^\nu + \frac{\partial_0 \phi \partial_1 \phi}{h_{11}} \partial_0 Z^\nu \right] = 0 \quad , \nu = 0, \dots, 3 \end{aligned} \quad (2.2)$$

$$\partial_0^2 \phi - \partial_1^2 \phi = 0. \quad (2.3)$$

## 2.4 Dynamical Circular Solutions

In reference [28], SPG discuss a circular superconducting cosmic string solution. The radius of the loop oscillates anharmonically and the loop bears a charge and current of equal magnitude. Copeland, Hindmarsh and Turok [26] also discuss a circular loop, with no charge, which they call a ‘spring’ because the current in the loop acts to oppose the string

tension and hence stabilizes the loop against collapse. Davis and Shellard [31], using a field theory approach, showed that charge can also stabilize a static loop against collapse and they call the resultant microscopic loops ‘vortons’. It will be shown here that charge can stabilize dynamic loops in exactly the same way as current and that the two perform essentially the same dynamic function, independently of each other. The next chapter will show that charges on superconducting cosmic strings may be large, and hence such charge-stabilized loops can be macroscopic.

The ansatz for the circular loop is

$$Z^\mu = \begin{bmatrix} t(\tau) \\ r(\tau)\mathbf{q}(\sigma) \end{bmatrix}. \quad (2.4)$$

The current and charge are induced by the scalar field  $\phi$ , assumed here to be of the form  $\phi = \kappa_0\tau + \kappa_1\sigma$ , where  $\kappa_0$  and  $\kappa_1$  are constants. This corresponds to direct current flow along the string and a uniform charge distribution. The magnitude of the current and charge may vary with time as the shape and size of the loop change.

The components of the world-sheet metric are

$$h_{00} = \dot{r}^2|\mathbf{q}|^2 - \dot{t}^2$$

$$h_{01} = r\dot{r}\mathbf{q}\cdot\mathbf{q}'$$



$$h_{11} = r^2 |\mathbf{q}'|^2 \quad (2.5)$$

where a dot denotes differentiation with respect to  $\tau$  and a dash with respect to  $\sigma$ . Applying the conformal gauge conditions restricts the form of  $\mathbf{q}$ . From  $h_{01} = 0$  we have  $\mathbf{q} \cdot \mathbf{q}' = 0$  and from  $h_{00} + h_{11} = 0$  we find that  $|\mathbf{q}'|$  is a constant, say  $L^{-1}$ . Also,  $|\mathbf{q}|$  must be a constant and it may be normalized to 1 by absorbing any factors into  $r$ . On differentiating  $\mathbf{q} \cdot \mathbf{q}'$  with respect to  $\sigma$ , one finds that  $\mathbf{q} \cdot \mathbf{q}'' = -L^{-2}$ .

Substituting into the equation of motion (2.2) gives, for the space components,

$$\begin{aligned} & \left( T + \frac{\kappa_0^2 + \kappa_1^2}{2r^2} L^2 \right) \ddot{\mathbf{r}} \mathbf{q} - \left( T - \frac{\kappa_0^2 + \kappa_1^2}{2r^2} L^2 \right) r \mathbf{q}'' \\ & - \frac{2\kappa_0 \kappa_1}{r} L^2 \dot{\mathbf{r}} \mathbf{q}' - \frac{\kappa_0^2 + \kappa_1^2}{r^3} L^2 \dot{\mathbf{r}}^2 \mathbf{q} + \frac{2\kappa_0 \kappa_1}{r^2} L^2 \dot{\mathbf{r}} \mathbf{q}' = 0. \end{aligned} \quad (2.6)$$

If we now contract this equation with  $\mathbf{q}$  we obtain

$$\begin{aligned} & \left( T + \frac{\kappa_0^2 + \kappa_1^2}{2r^2} L^2 \right) \ddot{\mathbf{r}} \cdot \mathbf{q} + \left( T - \frac{\kappa_0^2 + \kappa_1^2}{2r^2} L^2 \right) \frac{r}{L^2} \\ & - \frac{\kappa_0^2 + \kappa_1^2}{r^3} L^2 \dot{\mathbf{r}}^2 = 0. \end{aligned} \quad (2.7)$$

The substitution  $s = Tr - (\kappa_0^2 + \kappa_1^2)L^2/2r$  enables this to be greatly simplified because

$$\dot{s} = T\dot{r} + \frac{\kappa_0^2 + \kappa_1^2}{2r^2} L^2$$

and

$$\ddot{s} = \left( T + \frac{\kappa_0^2 + \kappa_1^2}{2r^2} L^2 \right) \ddot{r} - \frac{\kappa_0^2 + \kappa_1^2}{r^3} L^2 \dot{r}^2$$

and thus equation (2.7) may be written

$$\ddot{s} + \frac{s}{L^2} = 0, \quad (2.8)$$

which is anharmonic oscillation about the point  $r_0 = L\sqrt{(\kappa_0^2 + \kappa_1^2)/(2T)}$ .

The anharmonicity may be seen to be further complicated on consideration of the zeroth component of the equation of motion. The relationship between time and  $\tau$  is non-trivial:

$$\frac{d\tau}{dt} = K \left( T + \frac{\kappa_0^2 + \kappa_1^2}{2r^2} L^2 \right) \quad (2.9)$$

where  $K$  is a constant which may be calculated from the gauge conditions, (2.1).

The explicit form of the world-sheet two-current is given by

$$J^0 = q \frac{\partial_1 \phi}{\sqrt{-h}} = q \kappa_1 \frac{L^2}{r^2}$$

$$J^1 = -q \frac{\partial_0 \phi}{\sqrt{-h}} = -q \kappa_0 \frac{L^2}{r^2}.$$

The repulsive force which prevents the loop from collapsing to a point can be seen to result from both current and charge. The dynamical role of each

is identical. A superconducting cosmic string may form a stable loop with or without a net current and this possibility does not rely on the magnitude of the saturation current predicted by the theory. The implication is that stable loops must be considered a generic feature of superconducting string theories and not merely an inconvenient oddity that may arise in certain ranges of the parameters of the theory.

As a final remark for this section, it should be pointed out that although it has been tacitly assumed that  $\mathbf{q}(\sigma)$  is a radial vector tracing out a circle, this is not actually required. The gauge conditions  $\mathbf{q} \cdot \mathbf{q}' = 0$  and  $|\mathbf{q}'| = L^{-1}$  imply merely that  $\mathbf{q}$  is confined to the surface of a sphere. The actual shape of the string can be any closed curve inscribed on the surface of the sphere. The radius of that sphere varies anharmonically in the same manner as the radius of a circular loop.

## 2.5 Static Superconducting Cosmic Strings

As has been pointed out by CHHT [29], it is possible for a static string with arbitrary shape to be stabilized by current. The presence of current on the string produces a force which may be brought into exact balance with

the string tension, resulting in a net force of zero acting on the string. It may superficially appear difficult to conserve current on a current-stabilized string with varying radius of curvature, but it must be realized that the world-sheet current is a different entity to the space-time current and the relationship between the two depends on the curvature, allowing curvature effects to cancel out. Charge on the string may produce the same effect and here, again, as in the previous section, it is possible for any combination of charge and current to stabilize an arbitrarily shaped static string.

For a static string,  $\partial_0 Z^\nu = 0$  ,  $\nu = 1, 2, 3$  and  $\partial_1 Z^0 = 0$ . The gauge condition  $h_{00} + h_{11} = 0$  then requires that  $h_{11}$  is a constant. The equation of motion becomes, for the space components of  $Z$ ,

$$\left(T - \frac{\kappa_0^2 + \kappa_1^2}{2h_{11}}\right) \partial_1^2 Z^\nu = 0, \quad (2.10)$$

using the same solution for the current as before,  $\phi = \kappa_0 \tau + \kappa_1 \sigma$ . For a charge and/or current such that  $\kappa_0^2 + \kappa_1^2 = 2Th_{11}$ , the string shape is unrestricted. Static strings of arbitrary shape must also be considered generic features of superconducting string theories.

# Chapter 3

## Charge Separation

### 3.1 Introduction

It has been shown that a superconducting cosmic string bearing a current [29] may be stable against collapse and thus long-lived. The current needed to stabilize a string loop at a particular radius is related to the radius of the loop. Because current quenching provides an upper bound to the current which may be carried by a superconducting cosmic string, there is thus an upper bound to the radius of a loop which may be stabilized by current. For some theories this radius might be quite small and macroscopic loops would not form.

However, it has also been shown in reference [31] and in the previous chapter that charge can stabilize a string. Unlike current, charge is not limited by quenching. In fact, Davis and Shellard have shown [31] that a charged string actually has a greater capacity to carry current. A charge-stabilized string, then, may be much larger than one stabilized only by current, and must significantly affect cosmology.

It was demonstrated by Davis and Shellard [44] that, even in the absence of primordial magnetic fields, random fluctuations must produce currents and charge distributions capable of stabilizing microscopic strings, or vortons. The mechanism explored in the present work, by contrast, provides currents established by primordial magnetic fields the potential to create and maintain charges large enough to stabilize macroscopic strings.

## 3.2 Collisions

Charge separation may occur when superconducting cosmic strings collide because the centre-of-momentum reference frame of the collision will not in general be the same as the rest frames of the colliding strings. A string which may have no charge in its rest frame will in general possess a charge

distribution in some other frame. At the collision the strings intercommute [45] enabling the local charge distributions to produce nett global charges on the new strings formed after the collision.

The principle will be demonstrated here by means of an example which embodies all the relevant features. A circular loop, stabilized by current alone, collides with a long straight piece of string, which may be part of a larger loop.

The strings intersect at two points, as shown in figure 3.1. In order to avoid having to model the dynamics of kinked superconducting cosmic strings it is necessary to work in a frame where the two collisions are simultaneous. If the collisions are not simultaneous, leakage of charge will occur, which manifests itself in the lack of Lorentz invariance of the separated charge between the times of the collisions. At the time of collision, each string fragments at the intersection points then reconnects with the other string. The charge on each fragment will give the total charge on the new strings. It is, of course, possible for the strings to reconnect in such a way as to produce only one, larger, new string, but for the present purposes it will be assumed that two new strings are formed. Although some charge will likely be lost in the collision, this charge loss will be local to the inter-

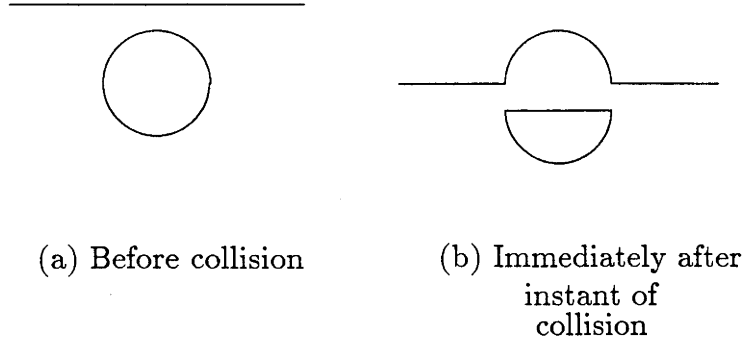


Figure 3.1: Configuration of the collision

section points and hence not greatly affect the total macroscopic charge for sufficiently long strings.

The rest-frame positions of the strings are

$$\begin{aligned}
 \mathbf{Z}_1(\sigma_1) &= (\alpha\sigma_1, 0, 0) \\
 \mathbf{Z}_2(\sigma_2) &= R(\cos \frac{\sigma_2}{R}, \sin \frac{\sigma_2}{R}, 0) \quad , \quad 0 \leq \sigma_2 \leq 2\pi R.
 \end{aligned} \tag{3.1}$$

They carry rest-frame current and charge densities

$$\begin{aligned}
 \rho_0^1(\mathbf{y}_0^1) &= 0 \\
 \mathbf{J}_0^1(\mathbf{y}_0^1) &= \int d\sigma_1 (\alpha j_1, 0, 0) \delta(y_{01}^1 - \alpha\sigma_1) \delta(y_{02}^1) \delta(y_{03}^1)
 \end{aligned}$$



$$\begin{aligned}
\rho_0^2(\mathbf{y}_0^2) &= 0 \\
\mathbf{J}_0^2(\mathbf{y}_0^2) &= \int_0^{2\pi R} d\sigma_2 j_2\left(-\sin \frac{\sigma_2}{R}, \cos \frac{\sigma_2}{R}, 0\right) \\
&\quad \delta(y_{01}^2 - R \cos \frac{\sigma_2}{R}) \delta(y_{02}^2 - R \sin \frac{\sigma_2}{R}) \delta(y_{03}^2). \quad (3.2)
\end{aligned}$$

Here  $y_{0i}^j$  refers to the  $i$ th component of a general position in the rest-frame of the  $j$ th string.

The collision is arranged by boosting each string separately to a common frame, the first by a velocity

$$\mathbf{v}^1 = (0, 0, v_3)$$

and the second by

$$\mathbf{v}^2 = (v_1, 0, 0)$$

with Lorentz factors

$$\gamma_1 = (1 - (v_3)^2)^{-\frac{1}{2}}$$

and

$$\gamma_2 = (1 - (v_1)^2)^{-\frac{1}{2}}.$$

In the frame of the collision, a general position  $\mathbf{y}_1$  may be written in terms of the rest frame positions as

$$\mathbf{y}_1 = (y_{01}^1, y_{02}^1, \gamma_1^{-1} y_{03}^1 - v_3 t)$$

$$= (\gamma_2^{-1} y_{02}^2 - v_1 t, y_{02}^2, y_{03}^2) \quad (3.3)$$

while the string positions are

$$\begin{aligned} \mathbf{x}_1(\sigma_1) &= (\alpha \sigma_1, 0, -v_3 t) \\ \mathbf{x}_2(\sigma_2) &= (\gamma_2^{-1} R \cos \frac{\sigma_2}{R} - v_1 t, R \sin \frac{\sigma_2}{R}, 0). \end{aligned} \quad (3.4)$$

Intersection occurs when  $t = 0$ ,  $\sigma_2 = 0, \pi R$  and  $\sigma_1 = \pm R/\alpha \gamma_2$ .

The boosted charges and currents become

$$\begin{aligned} \rho_1^1(\mathbf{y}_1) &= 0 \\ \mathbf{J}_1^1(\mathbf{y}_1) &= \int d\sigma_1 (\alpha j_1, 0, 0) \delta(y_{01}^1 - \alpha \sigma_1) \delta(y_{02}^1) \delta(\gamma_1^{-1} y_{03}^1 - v_3 t) \\ \rho_1^2(\mathbf{y}_1) &= \gamma_2 v_1 j_2 \int_0^{2\pi R} d\sigma_2 \sin \frac{\sigma_2}{R} \\ &\quad \delta(y_{01}^2 - R \cos \frac{\sigma_2}{R}) \delta(y_{02}^2 - R \sin \frac{\sigma_2}{R}) \delta(y_{03}^2) \\ \mathbf{J}_1^2(\mathbf{y}_1) &= \int_0^{2\pi R} d\sigma_2 j_2 (-\gamma_2 \sin \frac{\sigma_2}{R}, \cos \frac{\sigma_2}{R}, 0) \\ &\quad \delta(y_{01}^2 - R \cos \frac{\sigma_2}{R}) \delta(y_{02}^2 - R \sin \frac{\sigma_2}{R}) \delta(y_{03}^2). \end{aligned} \quad (3.5)$$

Now while there is no charge on the straight string, the circular (elliptical in this frame of reference) string has acquired a local charge distribution which becomes a global charge when the string fragments and intercommutes with the other string. This is illustrated in figure 3.2. The charge

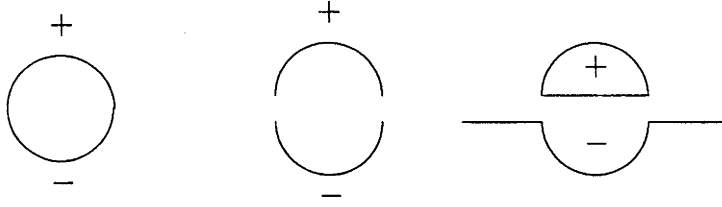


Figure 3.2: Charge is separated when the strings intercommute.

and current densities are now specified as functions of general position coordinates in the new common frame,  $y_1$ . The Jacobian

$$\begin{aligned}
 |J| &= \left| \frac{dy_1}{dy_0^2} \right| \\
 &= \begin{vmatrix} \gamma_2^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
 &= \gamma_2^{-1}
 \end{aligned} \tag{3.6}$$

enables integration of the total charge with respect to the new reference frame. The total charge separated is

$$\begin{aligned}
 Q_{sep} &= \int d^3 y_1 \rho_1^2 \\
 &= v_1 j_2 \int_0^{\pi R} d\sigma_2 \sin \frac{\sigma_2}{R} \\
 &= 2v_1 j_2 R.
 \end{aligned} \tag{3.7}$$

This separated charge is invariant under Lorentz boosts which preserve

the simultaneity of the collisions, but not under boosts which do not. If a further boost by a velocity

$$\mathbf{v}^3 = (0, v_{32}, v_{33})$$

with

$$\gamma_3 = (1 - (v_{32})^2 - (v_{33})^2)^{-\frac{1}{2}}$$

were applied to both strings, simultaneity of the collisions would be preserved because the collisions are separated by a displacement purely in the  $x$ -direction. A general position in this frame would be

$$\begin{aligned} y_{21} &= y_{11} \\ y_{22} &= \left( 1 - \gamma_3(v_{32})^2 + (\gamma_3 - 1) \frac{(v_{32})^2}{(v_{32})^2 + (v_{33})^2} \right) y_{12} \\ &\quad + v_{32}v_{33} \left( \frac{\gamma_3 - 1}{(v_{32})^2 + (v_{33})^2} - \gamma_3 \right) y_{13} \\ y_{23} &= v_{32}v_{33} \left( \frac{\gamma_3 - 1}{(v_{32})^2 + (v_{33})^2} - \gamma_3 \right) y_{12} \\ &\quad + \left( 1 - \gamma_3(v_{33})^2 + (\gamma_3 - 1) \frac{(v_{33})^2}{(v_{32})^2 + (v_{33})^2} \right) y_{13} \end{aligned} \tag{3.8}$$

The transformed charges would be

$$\begin{aligned} \rho_2^1(\mathbf{y}_2) &= 0 \\ \rho_2^2(\mathbf{y}_2) &= \gamma_3 \rho_1^2(\mathbf{y}_1) \end{aligned} \tag{3.9}$$

giving a separated charge

$$\begin{aligned}
Q'_{sep} &= \int d^3\mathbf{y}_2 \rho_2^2(\mathbf{y}_2) \\
&= \gamma_3 \int d^3\mathbf{y}_1 \left| \frac{\partial \mathbf{y}_2}{\partial \mathbf{y}_1} \right| \rho_1^2(\mathbf{y}_1) \\
&= Q_{sep}
\end{aligned}$$

since the Jacobian

$$\begin{aligned}
|J| &= |\partial \mathbf{y}_2 / \partial \mathbf{y}_1| \\
&= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 - \gamma_3(v_{32})^2 & v_{32}v_{33} \left( \frac{\gamma_3 - 1}{(v_{32})^2 + (v_{33})^2} - \gamma_3 \right) \\ 0 & v_{32}v_{33} \left( \frac{\gamma_3 - 1}{(v_{32})^2 + (v_{33})^2} - \gamma_3 \right) & 1 - \gamma_3(v_{33})^2 \\ & & + (\gamma_3 - 1) \frac{(v_{33})^2}{(v_{32})^2 + (v_{33})^2} \end{vmatrix} \\
&= \left( 1 - \gamma_3(v_{32})^2 + (\gamma_3 - 1) \frac{(v_{32})^2}{(v_{32})^2 + (v_{33})^2} \right) \\
&\quad \left( 1 - \gamma_3(v_{33})^2 + (\gamma_3 - 1) \frac{(v_{33})^2}{(v_{32})^2 + (v_{33})^2} \right) \\
&\quad - \left[ v_{32}v_{33} \left( \frac{\gamma_3}{(v_{32})^2 + (v_{33})^2} - \gamma_3 \right) \right]^2 \\
&= \gamma_3(1 - (v_{32})^2 - (v_{33})^2) \\
&= \gamma_3^{-1}.
\end{aligned} \tag{3.10}$$

However, if the boost was

$$\mathbf{v}^3 = (v_{31}, 0, 0)$$

with

$$\gamma_3 = (1 - (v_{31})^2)^{-\frac{1}{2}}$$

the collisions would no longer be simultaneous, but would be separated by a time difference  $\Delta t = 2R\gamma_3(\gamma_2)^{-1}$ . The transformed charges would be

$$\begin{aligned}\rho_2^1(\mathbf{y}_2) &= -\alpha\gamma_3v_{31}j_1 \int d\sigma_1 \delta(y_{01}^1 - \alpha\sigma_1) \delta(y_{02}^1) \delta(y_{03}^1) \\ \rho_2^2(\mathbf{y}_2) &= \gamma_3\gamma_2j_2(v_1 + v_{31}) \int_0^{2\pi R} d\sigma_2 \sin \frac{\sigma_2}{R} \\ &\quad \delta(y_{01}^2 - R \cos \frac{\sigma_2}{R}) \delta(y_{02}^2 - R \sin \frac{\sigma_2}{R}) \delta(y_{03}^2)\end{aligned}\quad (3.11)$$

giving a separated charge of

$$Q'_{sep} = 2Rj_2(v_1 + v_{31}) \pm \frac{2j_1Rv_{31}}{\gamma_1\gamma_2}. \quad (3.12)$$

This tells us that unless the collisions are simultaneous it is necessary to dynamically model the leakage of charge that occurs between the two collisions.

In order to see how large the invariant charge separation is, it is useful to compare it with the charge required to stabilize a circular string at the

radius of the colliding circular string, without current. It was demonstrated in chapter two that for the circular string stabilized by pure current in its rest-frame the magnitude of the current must be such that  $j_2 = q\sqrt{2T}$  where  $q$  is the charge of each of the charge carriers on the string and  $T$  is the string tension. It is found then that the separated charge is  $Q_{sep} = v_1 q R \sqrt{8T}$ .

To stabilize a loop of radius  $R$  with charge requires a charge density

$$\rho = \int_0^{2\pi R} d\sigma q \sqrt{2T} \delta^3(\mathbf{y} - \mathbf{Z}(\sigma))$$

giving a total charge

$$Q_{stab} = \int d^3\mathbf{y} \rho = \pi q R \sqrt{8T}. \quad (3.13)$$

The separated charge is thus

$$Q_{sep} = \frac{v_1}{\pi} Q_{stab}.$$

So, although the charge separated is not, in this case, large enough to stabilize the original string, it is clearly still quite large when the strings collide at a large relative velocity and would be large enough to stabilize a loop with radius  $v_1 R / \pi$ .

### 3.3 Conclusions

The evolution of a network of superconducting cosmic strings would thus involve large charges being continually exchanged. The electromagnetic properties of such a superconducting cosmic string network would be very complicated, and may not exhibit the simple features required for the original Ostriker, Thompson and Witten [25] explosive structure formation scenario.

Large charged stable loops may be formed after loop fragmentation has proceeded to the point where collisions are no longer common and hence most strings may be expected to retain their charge. The inertia of the charge carrying modes on the string would tend to round out structure. Arguing by analogy with the circular case, for a string with charge  $J^0$  and current  $|\mathbf{J}|$  locally, structure with a radius of curvature less than  $R = ((J^0)^2 + \mathbf{J}^2)/q\sqrt{2T}$  would be rounded out [46]. Cusps on ordinary cosmic strings can occur because the string tension equals the mass per unit length. For superconducting cosmic strings, however, the effective string tension is decreased by the presence of current or charge [29], so cusp formation would be suppressed. Emission of electromagnetic radiation would also tend to



smooth out structure on strings, so the tendency would be to form stable loops.

# Chapter 4

## Scalar Condensation

### 4.1 Introduction

The vacuum instability around a current-carrying superconducting cosmic string due to pair creation of  $W$  bosons has been discussed by Ambjorn, Nielsen and Olesen [47]. This chapter examines the vacuum instability around a charged superconducting cosmic string, because, as has been shown in chapters 2 and 3, superconducting cosmic strings may acquire large charges in collisions, and form mechanically stable trajectories with such large charges. It is shown here that scalar or pseudoscalar particles may form a charged condensate around a charged superconducting cos-

mic string. The discussion here uses the Klein-Gordon equation, which, strictly speaking, only models pure scalar particles. An exact treatment of pseudoscalar particles, such as pions, would need to account for the extra degrees of freedom provided by the internal structure of the compound particle. However, in a cosmological context, the scale of the structure is negligible and thus the Klein-Gordon equation may be used to provide an adequate treatment of pseudoscalar particles. Because it is the lightest known spin-0 particle, the pion would provide the most important contribution to a scalar condensate. The possibility of pion condensation has been studied extensively in the context of spherically-symmetric fields with finite-sized sources [48,49,50], but pion condensates do not appear to produce any important physical effects in those cases. It now seems that pion condensates can play an important role in the case of cosmic strings.

## 4.2 Electromagnetic potential of a superconducting cosmic string

Ambjorn, Nielsen and Olesen [47] treat an infinitely long, straight string with current,  $I$ , but no charge. They use an electromagnetic potential

$$\begin{aligned} A &= (0, 0, 0, A_3) \\ A_3 &= \frac{eI}{2\pi} \log \frac{r}{r_0} \end{aligned} \tag{4.1}$$

for arbitrary  $r_0$ .

However, for a very large current, gravitational effects may become significant. Moss and Poletti [51] have shown that a current may provide a significant contribution to the gravitational field. It is worth exploring whether gravity has a significant impact on the electromagnetic field.

From reference [51], the gravitationally-corrected potential is of the form

$$\frac{dA}{dr} = \frac{2I}{\alpha} e^{2\psi}.$$

They show that  $\alpha = \kappa r$  and  $\psi = -\log H$ , where

$$H = c_1 \left(\frac{r}{r_0}\right)^m + c_2 \left(\frac{r}{r_0}\right)^{-m},$$

and

$$m^2 = 16\pi \frac{GI^2}{\kappa^2}.$$

Here,  $r_0$  is the thickness of the string, and should not be confused with the  $r_0$  parameter used elsewhere in this chapter, which may take on arbitrary values. The constant  $\kappa$  is related to the angle deficit of the space-time around the string due to gravity and is not greatly different from 1. With a maximum current of the order of  $10^{20} A$  [24],  $m^2$  is restricted to a maximum of  $10^{-8}$ . The  $(r/r_0)^{\pm m}$  terms in  $H$  which could produce deviations from a logarithmic potential are thus negligible, as  $(r/r_0)^{\pm 10^{-4}} \sim 1$  very closely for a large range of values of  $r$ .

The logarithmic potential used in [47] is thus a good approximation. Although Moss and Poletti did not specifically address the problem of a charged superconducting cosmic string, it is unlikely that a similar argument would not produce the same conclusion, namely that a potential of the form

$$A_0 = \frac{e\rho}{2\pi} \log \frac{r}{r_0}$$

provides a good description of the field due to a charged superconducting cosmic string.

### 4.3 Boson condensation

There are two possibilities for the formation of bosonic condensates in a strong electric field. In one case [48], the attractive part of the effective potential enables anti-particle bound-states to form, and when the particle and antiparticle bound-state energy eigenvalues are identical, a particle pair can be produced at no energy cost. There is no immediate limit to the number of particles that may thus be produced, and since both particles of each pair are bound to the neighbourhood of the field source, a neutral condensate forms. Eventually, the rate at which particles and antiparticles collide and annihilate becomes large enough to compensate for particle creation due to the vacuum instability, and this provides an absolute upper bound to the condensate. It is likely that other dynamic effects would occur before this happened, however.

In the other case [49], the particle bound-state energies descend into the negative-energy continuum. If the particle bound-state energy is  $-m$ , the energy for pair production can be provided by sending the antiparticle to infinity. For bosons, there is no apriori limit to the number of pairs that may be produced this way, so the coulomb self-interaction must inevitably

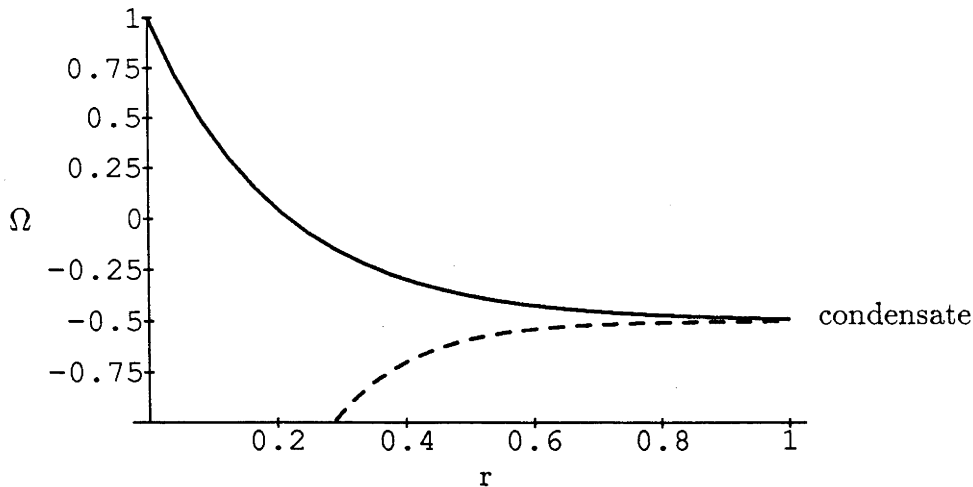


Figure 4.1: A neutral condensate: the particle (full curve) and antiparticle (dashed curve) energies meet, forming a neutral condensate.

become significant. The effect of the coulomb self-interaction is to screen the field of the string, and enough particles condense out of the vacuum to raise the bound-state energy levels above  $-m$ .

Klein and Rafelski [49], from their work with spherically symmetric potentials, identify the first case, the neutral condensate, with short-range potentials where  $\lim_{r \rightarrow \infty} rV(r) = 0$  and the second case, the charged condensate, with longer range potentials. The present work shows that in the case of cylindrical symmetry, the distinction is less clear-cut.

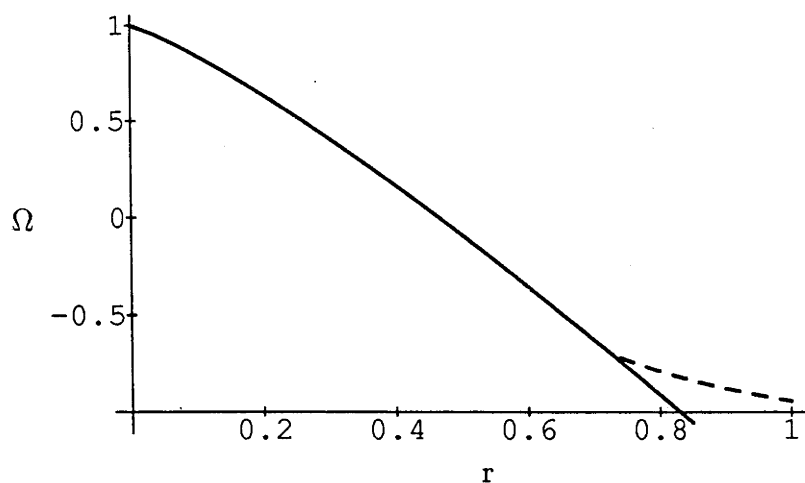


Figure 4.2: A charged condensate: the particle energies dive into the negative energy continuum, forming a charged condensate.

## 4.4 The wave equation for spin-zero particles in the field of a line-charge

The Klein-Gordon wave equation is

$$(D^\mu D_\mu - m^2)\phi = 0 \quad (4.2)$$

with

$$D_\mu = \partial_\mu - ieA_\mu$$

and

$$A_\mu = (A_0, \mathbf{0})$$



in the presence of a pure charge distribution. In order to avoid a logarithmic divergence at large distances, the potential is cut off at  $r = r_0$ , so

$$A_0 = \begin{cases} \frac{e\rho}{2\pi} \log \frac{r}{r_0}, & r \leq r_0 \\ 0, & r > r_0 \end{cases} \quad (4.3)$$

This is necessitated by the relativistic nature of the Klein-Gordon equation, which may be written as an effective Schrödinger equation with [50]

$$\begin{aligned} E_{\text{eff}} &= \frac{E^2 - m^2}{2}, \\ V_{\text{eff}} &= \frac{EV}{m} - \frac{V^2}{2m}. \end{aligned} \quad (4.4)$$

With  $V \sim \log r$ , the  $V^2$  term produces an infinite repulsion as  $r \rightarrow \infty$ , which is clearly unphysical and would prevent the existence of a ground-state eigenvalue. As  $r$  increases, the finite size of the string and the effect of screening would prevent the infinite repulsion from occurring. Thus,  $r_0$  is approximately the radius of the loop, although in a tightly-screened situation where the Debye screening due to the surrounding plasma is more significant than the effect of the finite size of the loop,  $r_0$  would more accurately represent the Debye length of the plasma. The imposition of a cut-off is not entirely physical either, but it will be shown later that it is quite a good approximation.

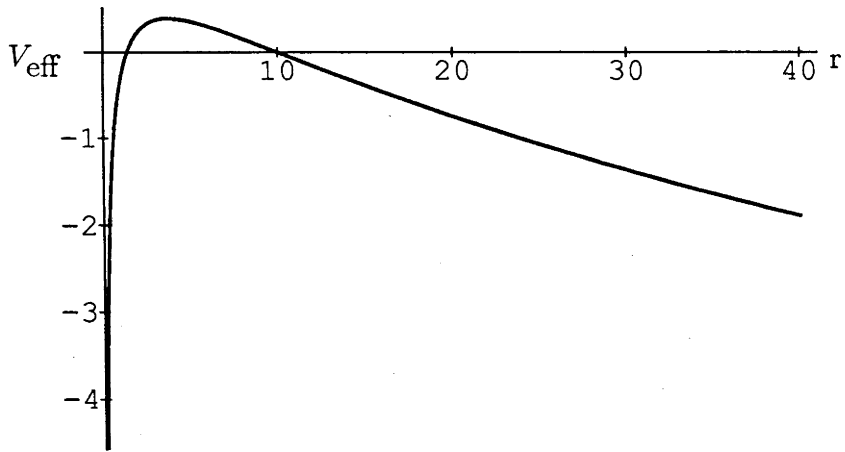


Figure 4.3: Effective potential with no cut-off.

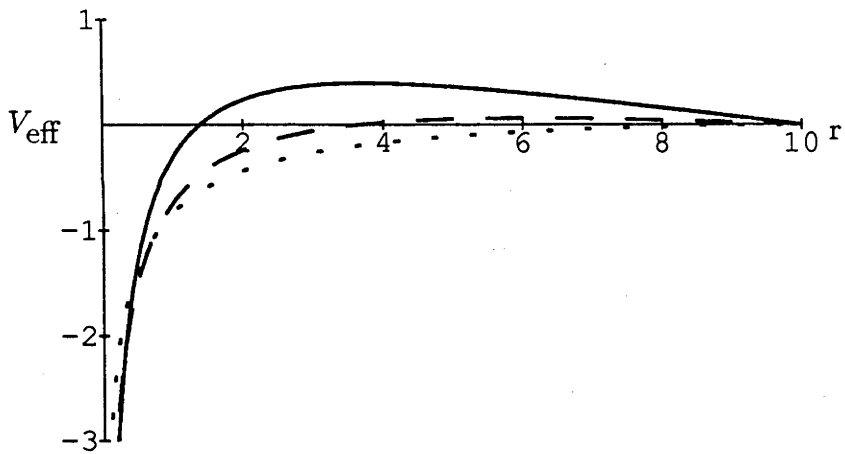


Figure 4.4: Cut off effective potentials for  $R_0 = 10$ ,  $\chi = 0.9$ , full curve,  $\chi = 0.7$ , dashed curve and  $\chi = 0.5$ , dotted curve.

The normalization of a Klein-Gordon wave function in the presence of the potential  $A^0$  and with energy eigenvalue  $\omega$  is [50]

$$N = 2 \int d^3x (\omega - A^0(x)) |\phi(x)|^2. \quad (4.5)$$

A positive normalization corresponds to a particle with energy  $\omega$ , while a negative normalization corresponds to an antiparticle with energy  $-\omega$ . The sign of the normalization of a state depends on the difference between the energy of the state and the value of the potential at each point, and hence on the shape of the potential.

As the potential has purely radial dependence, we can take

$$\phi = \phi(r) \exp(i\omega t + il\theta + ikz)$$

as an ansatz for the wave function. In cylindrical coordinates, the Klein-Gordon equation is then

$$\frac{d^2\phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} + \{(\omega - A_0)^2 - m^2 - k^2 - \frac{l^2}{r^2}\} \phi = 0. \quad (4.6)$$

It is clear that  $k$  only contributes to the effective mass and can therefore be absorbed into it. To facilitate computation, everything is scaled by the mass,  $\Omega = \omega/m$ ,  $R = rm$ ,  $\chi = e\rho/2\pi m$ , so the equation reads

$$\frac{d^2\phi}{dR^2} + \frac{1}{R} \frac{d\phi}{dR} + \{(\Omega - \frac{A_0}{m})^2 - 1 - \frac{l^2}{R^2}\} \phi = 0. \quad (4.7)$$

The first derivative term can be eliminated by letting

$$\phi = R^{-\frac{1}{2}}\psi,$$

then

$$\frac{d^2\psi}{dR^2} + \left\{ \left( \Omega - \frac{A_0}{m} \right)^2 - 1 + \frac{\frac{1}{4} - l^2}{R^2} \right\} \psi = 0. \quad (4.8)$$

## 4.5 Computation

The ground-state eigenvalues,  $\Omega$ , of the wave equation (4.8), were calculated using a shooting and matching technique. Asymptotic solutions for small and large  $R$  were evolved numerically using a fourth-order Runge-Kutta method, to meet at the cut-off point,  $R_0$ . An algorithm based on the secant method was used to adjust the trial eigenvalue to make the wave function smooth at the matching point.

For small  $R$ , a power series of the form

$$\psi = R^{l+\frac{1}{2}} \sum_{i,j=0}^3 a_{ij} R^i \log^j \frac{R}{R_0} \quad (4.9)$$

was fitted to the equation and the  $a_{ij}$ 's solved for by matching coefficients

to order  $R^{l+\frac{3}{2}}$ . For  $l = 0$ , the non-zero  $a_{ij}$ 's were

$$\begin{aligned}
a_{00} & \quad \text{arbitrary} \\
a_{20} &= -\frac{3}{8}\chi^2 - \frac{1}{2}\chi\Omega + \frac{1}{4}(1 - \Omega^2)a_{00} \\
a_{21} &= \frac{1}{2}\chi(\chi + \Omega)a_{00} \\
a_{22} &= -\frac{\chi^2}{4}a_{00}.
\end{aligned} \tag{4.10}$$

The large- $R$  asymptotic behaviour is

$$\psi \sim R^{-\frac{1}{2}} \exp(-\sqrt{1 - \Omega^2}R) \tag{4.11}$$

with the scale being a free parameter.

The groundstate was identified by counting the nodes in the wave function,  $\psi_L$ , as it was evolved outward from the small- $R$  asymptotic solution. The right-hand part of the wave function, evolved inwards from the large- $R$  asymptotic solution, is similarly denoted  $\psi_R$ . The part of the eigenvalue range  $-1 \leq \Omega \leq 1$  where the wave function had no nodes was subdivided into ten equal parts and the mismatch of the wave function,

$$\frac{1}{\psi_L} \frac{d\psi_L}{dR} - \frac{1}{\psi_R} \frac{d\psi_R}{dR},$$

was calculated at the eleven points thus identified. These points were

searched for a sign-change in the mismatch, and the two points where the sign-change occurred were used as starting points for the secant method.

The secant method is an iterative method for finding the zero of a function. The principle is to start with two points, one on either side of a sign-change in the function of which a zero is to be found. A straight line is constructed between the respective points on the graph, and the point where this line intersects with the axis is the new approximate location of the zero. Thus, if a function  $f(x)$  has values  $f_1$  at  $x = x_1$  and  $f_2$  at  $x = x_2$ , a new estimate of a zero of  $f$  is provided by

$$x = \frac{x_1 f_2 - x_2 f_1}{f_2 - f_1}.$$

This is illustrated in figure 4.5. The process is repeated by replacing one of the old  $x$ -values with the new one in such a way as to ensure that the two new points are still on either side of a sign-change. Iteration proceeds until a zero of the required accuracy is found.

The somewhat complicated front-end to the secant method was necessary because the eigenvalue was frequently to be found very close to the value that produced the first node, and thus produced a singularity and a sign-change in the mismatch. Restricting the search to the zero-node region

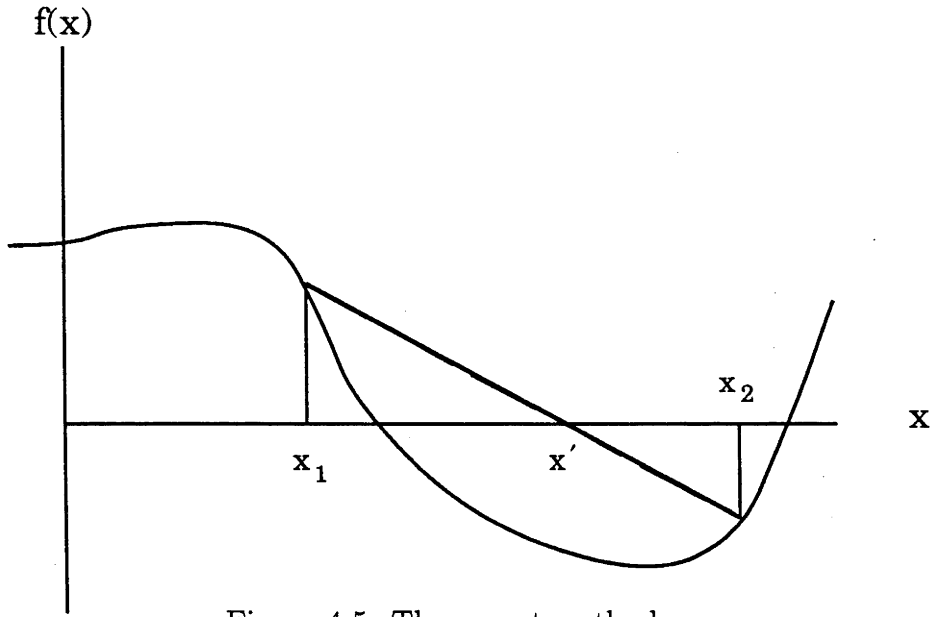


Figure 4.5: The secant method.

and performing a preliminary search to narrow down the region avoided the necessity of dealing directly with this singularity and compensated for the fact that the convergence of the secant method was often poor when the iterations were started far from the actual eigenvalue.

## 4.6 Results

Eigenvalues were calculated for varying field strengths, as shown in figure 4.6. For a sufficiently strong field, the eigenvalues dive down into the negative energy continuum, creating a charged condensate. To confirm this

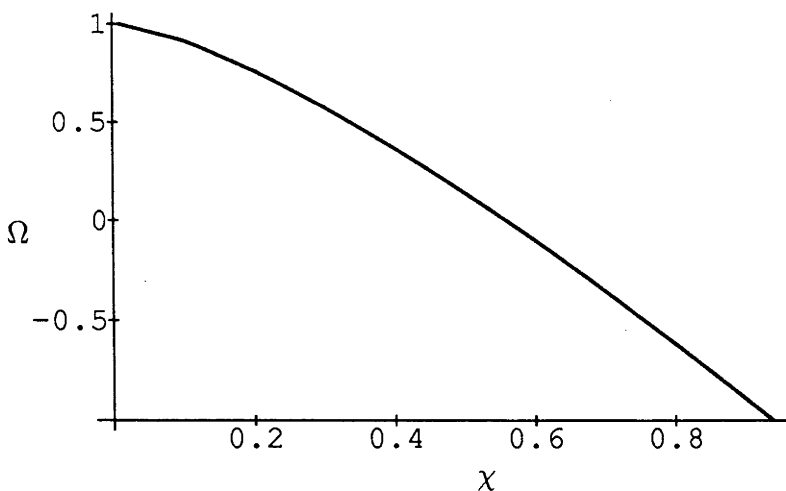


Figure 4.6: Eigenvalues for  $R_0 = 10$ .

conclusion, the normalization of the wave functions was calculated and found to be positive. The states were thus particle states and hence the condensate formed would be charged, not neutral.

The logarithmic potential falls off quite slowly, and were it not cut off one would expect it to behave like a ‘long-range’ potential. With a cut-off, one might expect it to behave more like a ‘short-range’ potential, as discussed in references [48] and [49]. However, the behaviour exhibited by the cut-off logarithmic potential is like that of a ‘long-range’ potential. It seems that the rigid classification of potentials as short- or long-ranged needs to be more flexible in the case of cylindrical symmetry.

Sample eigenfunctions are shown in figures 4.7-4.12. The dependence



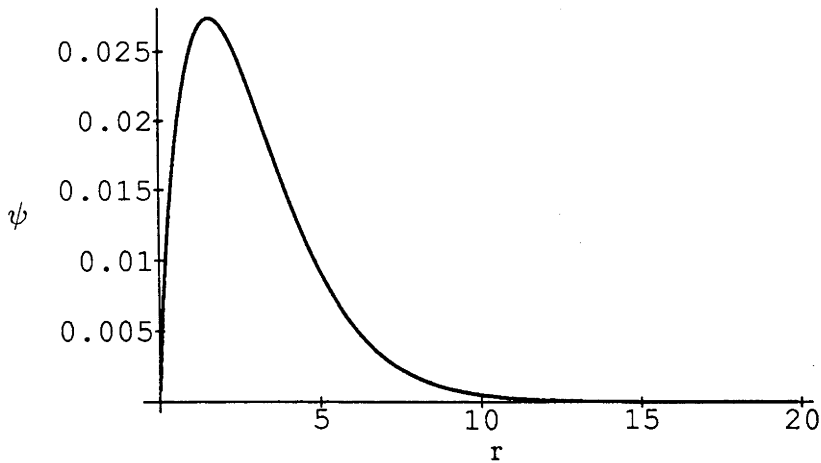


Figure 4.7: Eigenfunction for  $R_0 = 10$ ,  $\chi = 0.2$ .

of the eigenvalues on the cut-off radius was tested to make sure the cut-off didn't overtly interfere with the physics. Changing  $R_0$  should, if the cut-off approximation were exact, result merely in a simple shift of the eigenvalues. This is almost exactly what figure 4.13 demonstrates, with some slight deviation as the field-strength increases. The reason for the deviation with large  $\chi$  can be seen in figure 4.4, where as  $\chi$  increases, a narrowing bulge in the effective potential forms and tunnelling through that bulge would become more and more important. This effect is clearly quite small, however, and the use of the cut-off in the potential is thus vindicated.

Because increasing  $R_0$  shifts the eigenvalues downwards, a larger (or

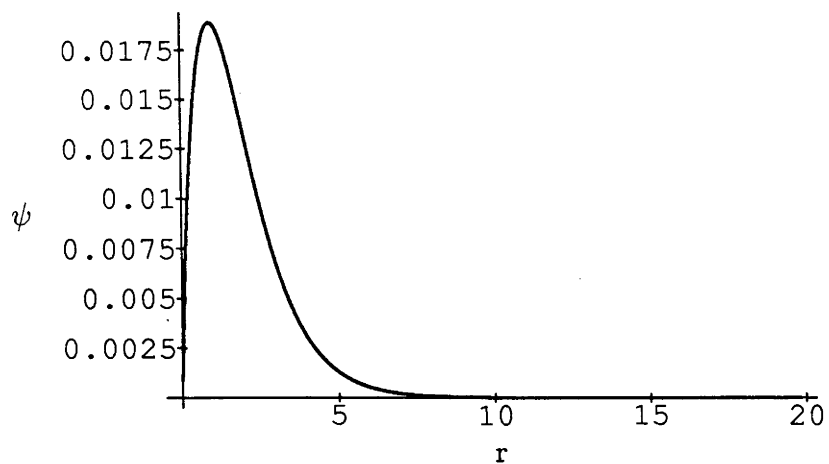


Figure 4.8: Eigenfunction for  $R_0 = 10$ ,  $\chi = 0.5$ .

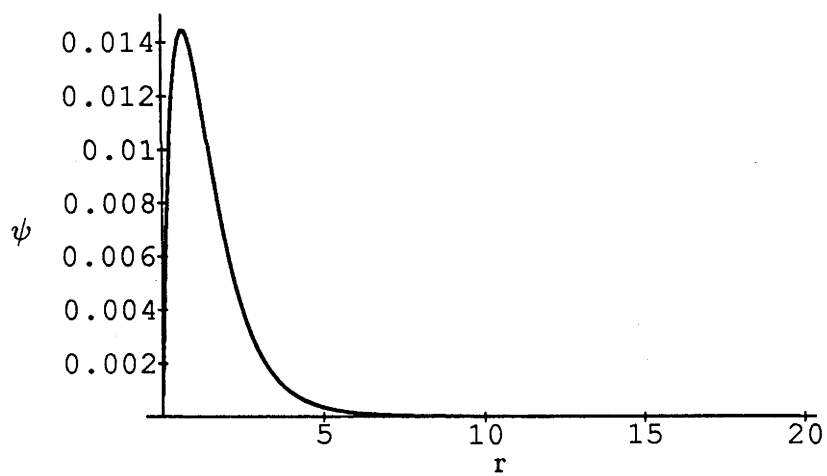


Figure 4.9: Eigenfunction for  $R_0 = 10$ ,  $\chi = 0.9$ .

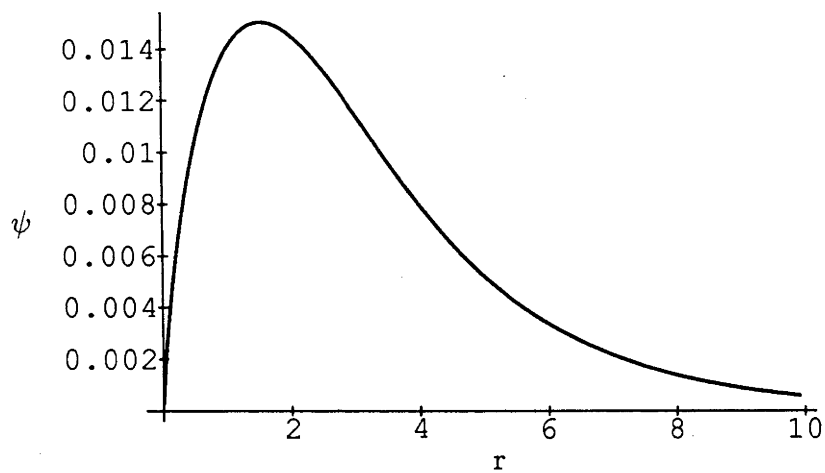


Figure 4.10: Eigenfunction for  $R_0 = 5$ ,  $\chi = 0.2$ .

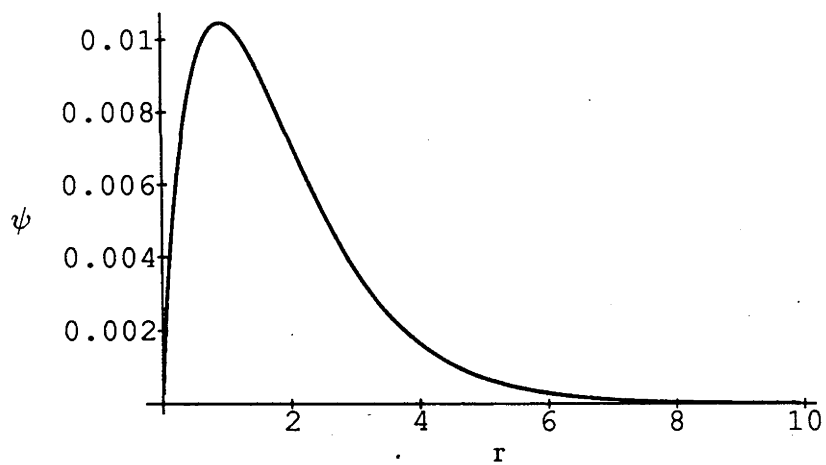


Figure 4.11: Eigenfunction for  $R_0 = 5$ ,  $\chi = 0.5$ .

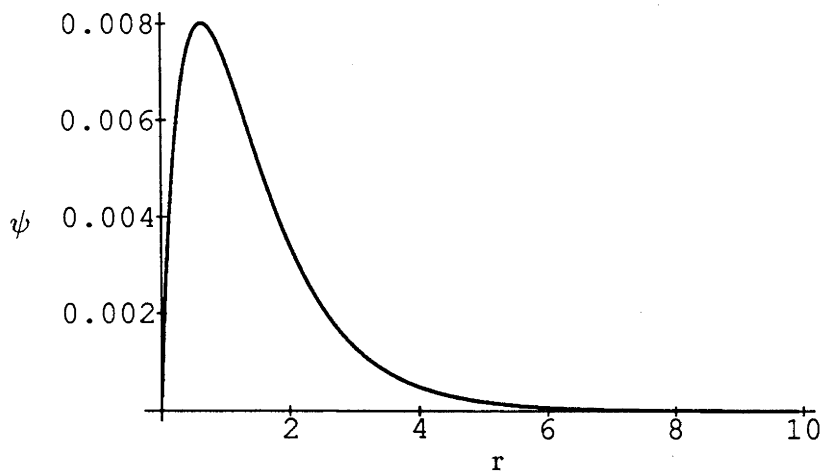


Figure 4.12: Eigenfunction for  $R_0 = 5$ ,  $\chi = 0.9$ .

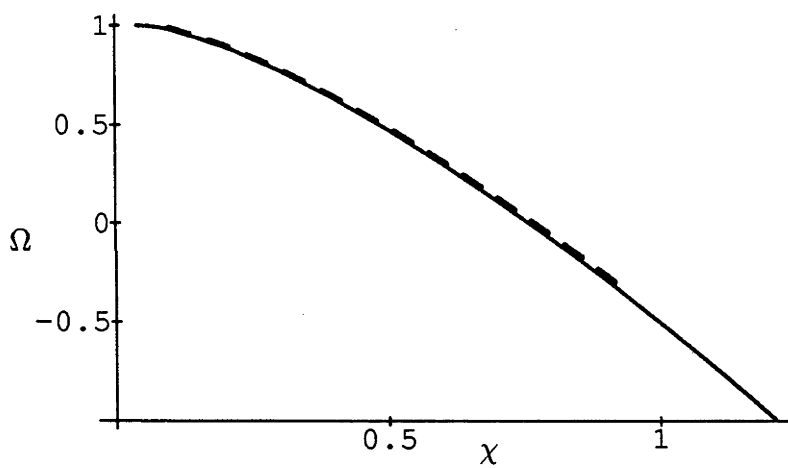


Figure 4.13: Comparison of eigenvalues. Eigenvalues for  $R_0 = 10$  (dashed curve) are shifted by  $\chi \log 2$  and superimposed onto eigenvalues of  $R_0 = 5$  (full curve).

more loosely screened) string would require a smaller charge density to produce a condensate. The maximum apparent charge density would thus be larger on smaller string loops, so electromagnetic effects may become more important as a network of cosmic strings breaks up into smaller and smaller loops. The collisional charge-separation mechanism discussed in chapter 3 also contributes to this tendency.

## 4.7 Conclusions

A charged superconducting cosmic string may evidently be surrounded by a cloud of charged scalar particles, most probably pions. The maximum apparent charge of such a string would be limited because of screening by this condensate. From a simple extrapolation of the results presented here, a charged string of radius 300 kiloparsecs, such as the one that Mathewson [52] suggests may be responsible for the Great Attractor, would have a maximum apparent charge of the order of  $10^{14}$  electronic charges per centimetre. A charge comparable with Witten's estimate of a  $10^{20}A$  current [24], which would correspond to a charge density of the order of  $10^{28}$  electron charges per centimetre, would evidently be very strongly shielded by

the condensate, and would not be visible to the rest of the universe.

If the condensate consists of pions, there is obviously the possibility of pion decay. This would be complicated, however, by the fact that the decay products (excluding neutrinos) would also experience the electric field. The next chapter discusses how spin- $\frac{1}{2}$  particles interact with an intense logarithmic electric field, so a discussion of the behaviour of pion decay products is delayed until then.

# Chapter 5

## Spin- $\frac{1}{2}$ Vacuum Instability

### 5.1 Introduction

A vacuum instability may also be caused by charged spin- $\frac{1}{2}$  particles and, although the effects may be less spectacular than in the bosonic case because of the Pauli exclusion principle, it would happen more often because electrons are much lighter than pions. This chapter explores the possibility of a spin- $\frac{1}{2}$  vacuum instability in the field of a charged superconducting cosmic string.

## 5.2 Vacuum instability of Dirac particles

Vacuum instability in the Dirac case is different from the bosonic case in two important ways. The Pauli exclusion principle prevents formation of a condensate and the phenomenon where a bound anti-particle state forms and meets the particle groundstate does not occur. The neutral condensate from the bosonic case thus has no counterpart in the Dirac case, but the charged condensate does, albeit significantly modified due to the exclusion principle.

As the intensity of the field is increased, the groundstate energy eigenvalue becomes more negative and eventually dives down into the negative energy continuum [50]. The eigenstate becomes an anti-resonance of the vacuum. If unoccupied, it represents a ‘hole’ in the negative energy sea, and thus an antiparticle. Having this eigenstate unoccupied is clearly unstable because the energy liberated by putting a particle there would be enough to create the particle, so ultimately the vacuum decays by emission of the antiparticle and absorption of a particle. The stable vacuum state is therefore doubly charged.

If the intensity of the source field is increased still further, other energy



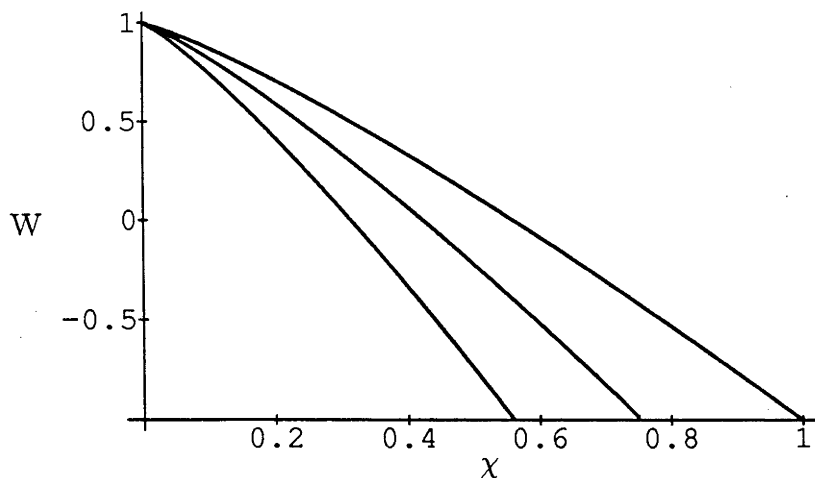


Figure 5.1: Energy states dive into the negative continuum as the field intensity increases.

eigenstates may descend into the negative continuum. Each state that dives will increase the vacuum charge by two. The density of energy states would thus determine to what extent the source charge is screened by the charged vacuum. If the vacuum charge increases as fast as, or faster than, the source charge, there will be an upper limit to the apparent charge of the source. If not, screening will decrease the apparent charge but not impose an upper bound.

### 5.3 Wave equation for spin- $\frac{1}{2}$ particles in the field of a line charge

Because the system possesses cylindrical symmetry, the  $z$ -component of angular momentum,  $m_j$ , is a good quantum number. The eigenvalue equation for  $m_j$  thus holds;

$$\left(\frac{1}{i}\frac{\partial}{\partial\phi} + \frac{1}{2}\sigma_z\right)\Psi = m_j\Psi \quad (5.1)$$

with

$$\sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (5.2)$$

and

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \quad (5.3)$$

So

$$\begin{aligned} \left(\frac{1}{i}\frac{\partial}{\partial\phi} + \frac{1}{2}\right)\psi_{1,3} &= m_j\psi_{1,3} \\ \left(\frac{1}{i}\frac{\partial}{\partial\phi} - \frac{1}{2}\right)\psi_{2,4} &= m_j\psi_{2,4} \end{aligned} \quad (5.4)$$

and therefore

$$\Psi = \begin{pmatrix} R_1(r)e^{i(m_j-\frac{1}{2})\phi} \\ R_2(r)e^{i(m_j+\frac{1}{2})\phi} \\ R_3(r)e^{i(m_j-\frac{1}{2})\phi} \\ R_4(r)e^{i(m_j+\frac{1}{2})\phi} \end{pmatrix} e^{ikz}. \quad (5.5)$$

The Dirac equation may be written [53]

$$W\Psi = (V + \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m)\Psi \quad (5.6)$$

where

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (5.7)$$

$$\begin{aligned} \alpha \cdot p &= \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 \\ &= \begin{pmatrix} 0 & 0 & p_3 & p_1 - ip_2 \\ 0 & 0 & p_1 + ip_2 & -p_3 \\ p_3 & p_1 - ip_2 & 0 & 0 \\ p_1 + ip_2 & -p_3 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (5.8)$$

Now in cartesian coordinates,  $p_1 = \frac{1}{i} \frac{\partial}{\partial x}$ ,  $p_2 = \frac{1}{i} \frac{\partial}{\partial y}$  and  $p_3 = \frac{1}{i} \frac{\partial}{\partial z}$ . However, cylindrical coordinate versions are more useful in the present case. In cylindrical coordinates,  $x = r \cos \phi$ ,  $y = r \sin \phi$  and  $z = z$ .

$$\begin{aligned} \frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi} \frac{\partial}{\partial y} \\ &= -r \sin \phi \frac{\partial}{\partial x} + r \cos \phi \frac{\partial}{\partial y} \\ \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} \\ &= \cos \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial y}. \end{aligned}$$

Solving for  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial y}$  gives

$$\frac{\partial}{\partial x} = \cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin \phi \frac{\partial}{\partial r} + \frac{1}{r} \cos \phi \frac{\partial}{\partial \phi}$$

and thus

$$\begin{aligned} p_1 + ip_2 &= -ie^{i\phi} \left( \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \phi} \right) \\ p_1 - ip_2 &= -ie^{-i\phi} \left( \frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \phi} \right). \end{aligned} \quad (5.9)$$

Substituting these into (5.6) gives

$$\begin{aligned} (W - V - m)R_1 &= kR_3 - i \left( \frac{d}{dr} + \frac{m_j + \frac{1}{2}}{r} \right) R_4 \\ (W - V - m)R_2 &= -i \left( \frac{d}{dr} - \frac{m_j - \frac{1}{2}}{r} \right) R_3 - kR_4 \\ (W - V + m)R_3 &= kR_1 - i \left( \frac{d}{dr} + \frac{m_j + \frac{1}{2}}{r} \right) R_2 \\ (W - V + m)R_4 &= -i \left( \frac{d}{dr} - \frac{m_j - \frac{1}{2}}{r} \right) R_1 - kR_2. \end{aligned} \quad (5.10)$$

For  $k = 0$ , the four equations decouple into two sets of two equations,

$$\begin{aligned} (W - V - m)R_1 &= -i \left( \frac{d}{dr} + \frac{m_j + \frac{1}{2}}{r} \right) R_4 \\ (W - V + m)R_4 &= -i \left( \frac{d}{dr} - \frac{m_j - \frac{1}{2}}{r} \right) R_1 \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} (W - V - m)R_2 &= -i \left( \frac{d}{dr} - \frac{m_j - \frac{1}{2}}{r} \right) R_3 \\ (W - V + m)R_3 &= -i \left( \frac{d}{dr} + \frac{m_j + \frac{1}{2}}{r} \right) R_2. \end{aligned} \quad (5.12)$$

Replacing  $m_j$  by  $-m_j$  transforms the second pair of equations into the first, so we need only consider (5.11).

For  $k \neq 0$ , the equations may be solved by letting

$$\begin{aligned} R_2 &= \frac{k}{\sqrt{m^2 + k^2} + m} R_4 \\ R_3 &= \frac{k}{\sqrt{m^2 + k^2} + m} R_1, \end{aligned} \quad (5.13)$$

which reduces both pairs of equations to

$$\begin{aligned} (W - V - \sqrt{m^2 + k^2})R_1 &= -i\left(\frac{d}{dr} + \frac{m_j + \frac{1}{2}}{r}\right)R_4 \\ (W - V + \sqrt{m^2 + k^2})R_4 &= -i\left(\frac{d}{dr} - \frac{m_j - \frac{1}{2}}{r}\right)R_1. \end{aligned} \quad (5.14)$$

These are identical to (5.11) except that the mass,  $m$ , is replaced by  $\sqrt{m^2 + k^2}$ . Thus,  $k$  merely increases the effective mass of the particles. For convenience,  $k$  will be absorbed into  $m$  in the calculations which follow.

The factor of  $i$  may be eliminated by replacing  $R_4$  with  $R_4/i$ . Computation is also facilitated by scaling all quantities by the effective mass,  $m$ , and thus replacing  $r$  by  $r/m$ ,  $W$  by  $mW$  and  $V$  by  $mV$ . To avoid any possibility of confusion with the wave functions,  $R_i$ , the convention of the previous chapter, where scaled quantities were capitalized, is not adopted here.

The equations used to find eigenvalues of the Dirac equation were thus

$$(W - V - 1)R_1 + \left(\frac{d}{dr} + \frac{m_j + \frac{1}{2}}{r}\right)R_4 = 0 \quad (5.15)$$

$$(W - V + 1)R_4 - \left(\frac{d}{dr} - \frac{m_j - \frac{1}{2}}{r}\right)R_1 = 0. \quad (5.16)$$

The potential is the same as that used in the scalar case,

$$V = \begin{cases} \chi \log \frac{r}{r_0}, & r < r_0 \\ 0, & r \geq r_0 \end{cases} \quad (5.17)$$

with

$$\chi = \frac{e\rho}{2\pi m}.$$

The necessity for the cut-off may be seen by considering the effective potential obtained by transforming (5.15) and (5.16) into an effective Schrödinger equation.

Defining

$$\alpha = W - V + 1$$

and

$$\beta = W - V - 1$$

and combining (5.15) with (5.16) gives

$$\left[ \frac{d^2}{dr^2} + \alpha\beta + \frac{1}{r} \frac{d}{dr} - \frac{\alpha'}{\alpha} \frac{d}{dr} - \frac{m_j^2 - m_j + \frac{1}{4}}{r} + \frac{\alpha' m_j - \frac{1}{2}}{\alpha r} \right] R_1 = 0. \quad (5.18)$$

The first derivative terms may be eliminated by the substitution

$$R_1 = \alpha^{\frac{1}{2}} r^{-\frac{1}{2}} G(r)$$

giving an equation

$$\left[ \frac{d^2}{dr^2} + \alpha\beta - \frac{m_j(m_j - 1)}{r^2} + \frac{m_j}{r} \frac{\alpha'}{\alpha} - \frac{3}{4} \left( \frac{\alpha'}{\alpha} \right)^2 + \frac{1}{2} \frac{\alpha''}{\alpha} \right] G = 0, \quad (5.19)$$

which may be written as [54]

$$\left[ \frac{d^2}{dr^2} + W^2 - 1 - \frac{m_j(m_j + 1)}{r^2} - U_{m_j}(r) \right] G = 0 \quad (5.20)$$

with

$$U_{m_j}(r) = 2WV - V^2 - \frac{m_j}{r} \frac{\alpha'}{\alpha} + \frac{3}{4} \left( \frac{\alpha'}{\alpha} \right)^2 - \frac{1}{2} \frac{\alpha''}{\alpha}. \quad (5.21)$$

The  $-V^2$  term also occurred in the Klein-Gordon case. It has the same effect here, which is to produce an infinite repulsion. This again requires a cut-off to enable bound states to form. There are also problems with the  $\alpha'/\alpha$  and  $\alpha''/\alpha$  terms when  $\alpha = 0$ , but this happens outside the cut-off. Using the physical interpretation of  $r_0$  as the radius of the loop (or the debye length of the plasma if it is smaller), one can see that such behaviour is unlikely to have a physical basis.

The short-distance behaviour of the effective potential also merits examination, as short-distance singularities can be troublesome for the Dirac



equation. With  $V = \chi \log \frac{r}{r_0}$ ,  $\alpha = W + 1 - \chi \log \frac{r}{r_0}$ ,  $\alpha' = -\frac{\chi}{r}$  and  $\alpha'' = \frac{\chi}{r^2}$ .

For small  $r$ ,  $\alpha \approx -\chi \log \frac{r}{r_0}$  and

$$U_{m_j} \approx 2WV - V^2 - \frac{m_j - \frac{1}{2}}{r^2 \log r} + \frac{3}{4} \frac{1}{r^2 (\log r)^2}. \quad (5.22)$$

The last term presents no problems because it is repulsive, despite being singular at  $r = 0$ . The third term is attractive for  $m_j \geq \frac{3}{2}$  but its effect would be swamped by the repulsive centrifugal term,  $\frac{m_j(m_j+1)}{r^2}$ . There is thus no short-distance problem in this case.

## 5.4 Computation

The computational procedure used in the Klein-Gordon case was found to work well, with minor modifications, in the Dirac case also. The same fourth-order Runge-Kutta algorithm was used to evolve the wave equations and the secant method was again used to match the wave functions. Because the Dirac equation is first-order, unlike the second-order Klein-Gordon equation, it was necessary only to match the wavefunctions rather than their derivatives. As there are effectively two wavefunctions in this case, matching both wavefunctions is computationally equivalent to matching the single Klein-Gordon wavefunction and its derivative. The actual

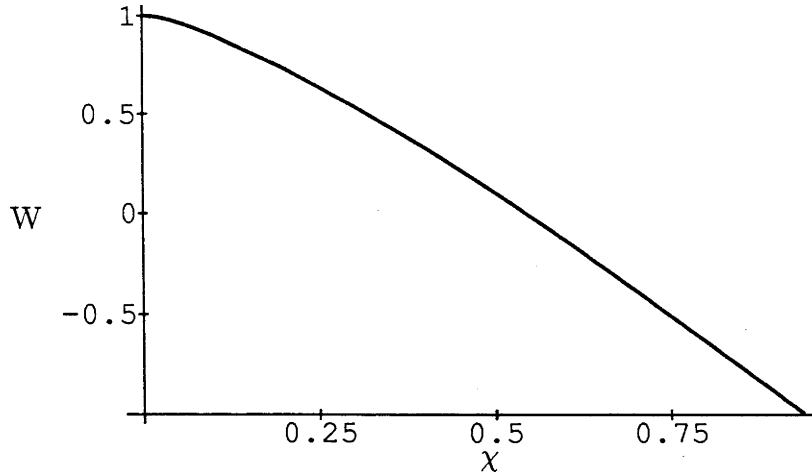


Figure 5.2: Eigenvalues calculated for  $r_0 = 10$ ,  $k = 0$ ,  $m_j = \frac{1}{2}$

function matched, then, was

$$\frac{R_{4L}}{R_{1L}} - \frac{R_{4R}}{R_{1R}}.$$

## 5.5 Results

Eigenvalues were calculated for varying field strengths, as figure 5.2 shows. It was found that for a sufficiently intense field, the eigenvalues would dive down into the negative energy continuum, causing a vacuum instability. A comparison of eigenvalues for different values of the cut-off radius,  $r_0$ , was made and is displayed in figure 5.9. Sample eigenfunctions are shown in figures 5.3-5.8.

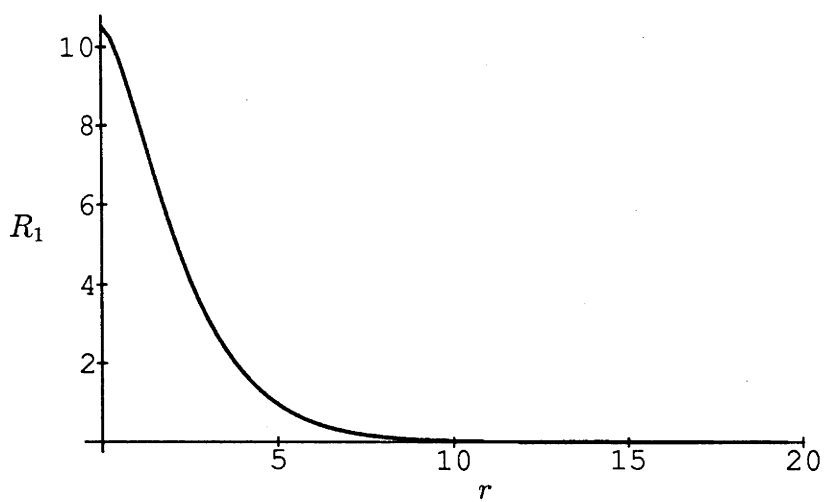


Figure 5.3: Eigenfunction  $R_1$  for  $\chi = 0.2$ ,  $r_0 = 10$ ,  $k = 0$ ,  $m_j = \frac{1}{2}$

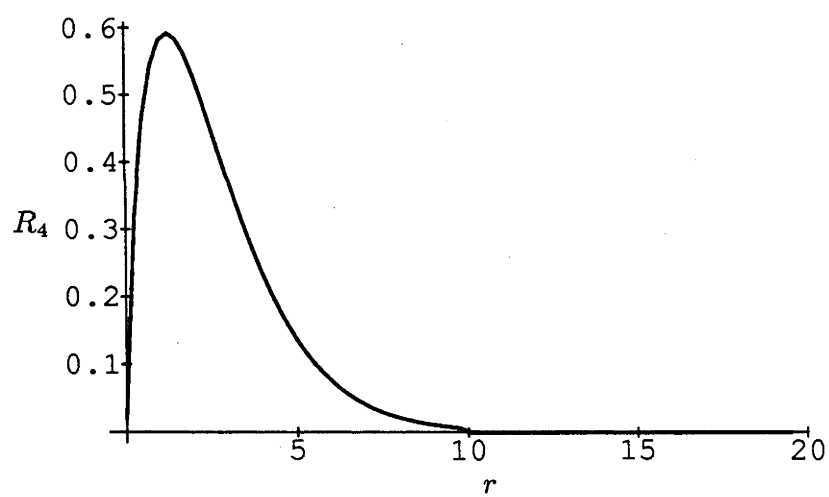


Figure 5.4: Eigenfunction  $R_4$  for  $\chi = 0.2$ ,  $r_0 = 10$ ,  $k = 0$ ,  $m_j = \frac{1}{2}$

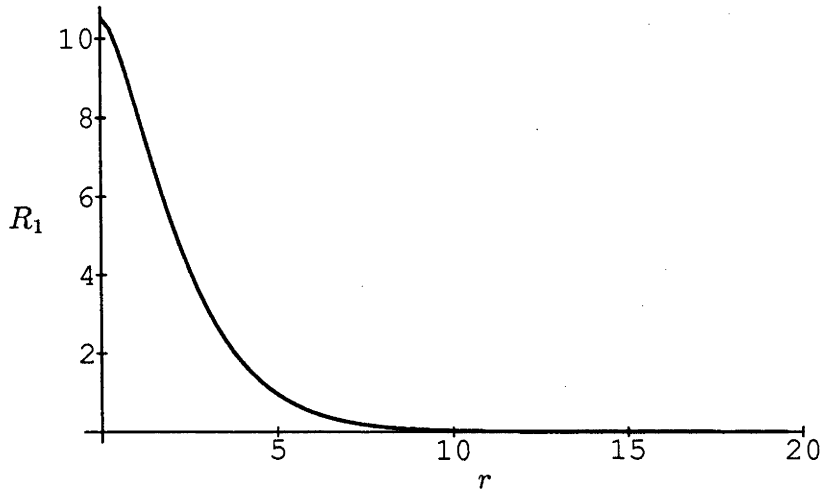


Figure 5.5: Eigenfunction  $R_1$  for  $\chi = 0.5$ ,  $r_0 = 10$ ,  $k = 0$ ,  $m_j = \frac{1}{2}$

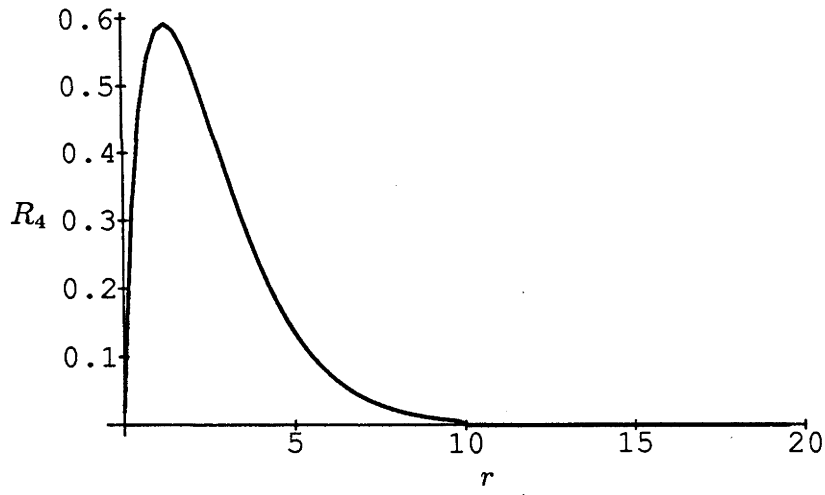


Figure 5.6: Eigenfunction  $R_4$  for  $\chi = 0.5$ ,  $r_0 = 10$ ,  $k = 0$ ,  $m_j = \frac{1}{2}$

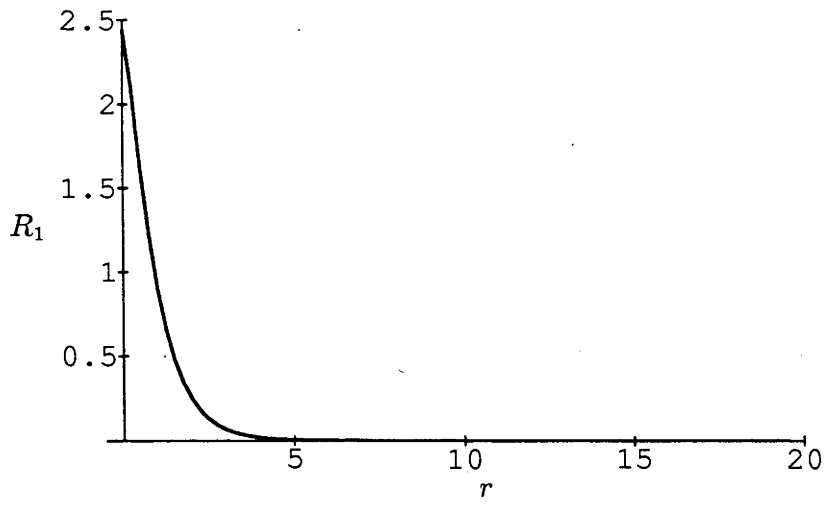


Figure 5.7: Eigenfunction  $R_1$  for  $\chi = 0.9$ ,  $r_0 = 10$ ,  $k = 0$ ,  $m_j = \frac{1}{2}$

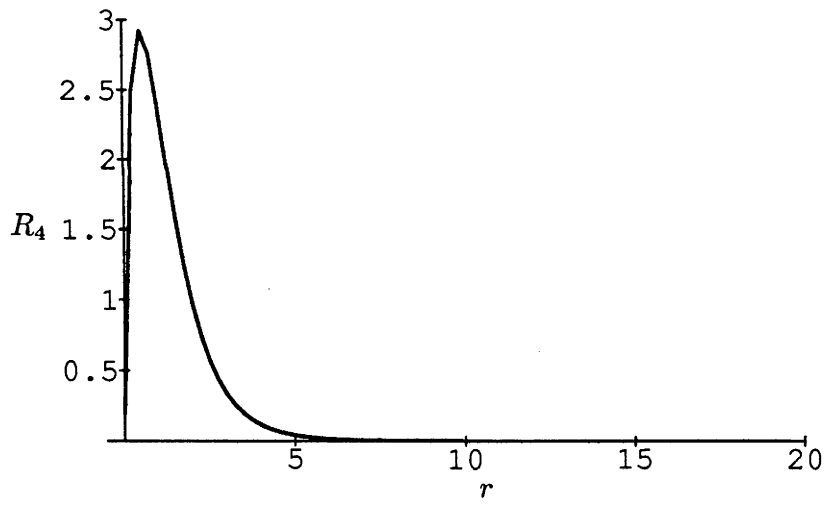


Figure 5.8: Eigenfunction  $R_4$  for  $\chi = 0.9$ ,  $r_0 = 10$ ,  $k = 0$ ,  $m_j = \frac{1}{2}$

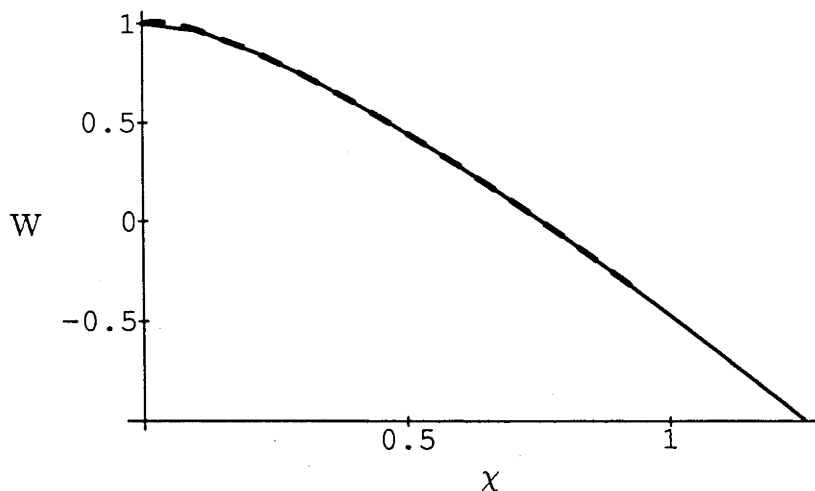


Figure 5.9: Comparison of shifted eigenvalues for  $r_0 = 10$  (dashed curve) with eigenvalues for  $R_0 = 5$  (full curve).

## 5.6 Conclusions

It is clear that a vacuum instability may be induced by spin- $\frac{1}{2}$  particles in the field of a charged superconducting cosmic string. Electrons, being lighter than pions, would cause an instability for a much smaller charge-density, but this does not imply that a fermionic instability would be more significant. Because each energy eigenstate that dives into the negative continuum causes only one electron-positron pair to condense out of the vacuum, the density of higher energy states determines how many particles would form around the string.

To examine higher eigenstates, one would have to find energy eigen-

values for  $k \neq 0$ . The quantization of  $k$  would depend on the length of the string. Longer strings would thus have a higher density of states than small strings. Further work is required to ascertain exactly how significant the fermionic instability would be compared with the bosonic condensate. Certainly, the fermions produced by the vacuum would screen the charge of the string and inhibit the formation of the pion condensate, but to what extent it is difficult to say authoritatively.

The fermionic charge of the vacuum would also influence the pion condensate more directly, because the fermionic products of pion decay would feel the field of the charged string. The pions would decay first into muons and neutrinos, and thence into electrons. A field intense enough to produce a pion condensate would, however, be intense enough to cause the vacuum to produce electrons and muons in the neighbourhood of the string, so Pauli exclusion pressure would eventually tend to inhibit decay of muons into electrons, and then inhibit pion decay itself. The details of these processes would be an interesting avenue of future exploration, but are beyond the scope of the present work.

# Chapter 6

## W Boson Condensation

### 6.1 Introduction

Ambjorn, Nielsen and Olesen (ANO) [47] explored the vacuum instability around an uncharged, current-carrying string due to spin-one W bosons. The possibility of W condensation around a charged string also merits consideration, though the effect may be swamped by pion condensation. An approximate analytic treatment has been undertaken by Shi and Li [55], who conclude that a W condensate may form around a string with both charge and current. Because, as they point out, their gaussian wave-packet approximation is not valid at short distances, the approach used in the



present work is more reliable. It will be demonstrated, using the results of Chapter 4, that a condensate similar in nature to the pion condensate may form, but, as will be seen, a complete exploration of the possibilities of W boson condensation is much more difficult using the techniques described here.

## 6.2 Wave equations

The derivation of the wave equations for W bosons in the field of a charged superconducting cosmic string follows analogously to that of ANO [47] for a current-bearing string. The linearized wave equations for a general background field are

$$D^\nu D_\nu W_\mu - D_\mu D_\nu W^\nu - m_W^2 W_\mu - 2ieF_{\mu\nu}W^\nu = 0. \quad (6.1)$$

The wave function also satisfies

$$m_W^2 D_\mu W^\mu = ieJ_\mu W^\mu \quad (6.2)$$

with a current  $J_\mu$  as a source, which implies that as long as  $J_\mu W^\mu = 0$  is true, one can assume  $D_\mu W^\mu = 0$  and simplify equation (6.1) to

$$D^\nu D_\nu W_\mu - m_W^2 W_\mu - 2ieF_{\mu\nu}W^\nu = 0. \quad (6.3)$$

For a string with charge but no current, the vector potential has the form

$$A = (A^0, 0, 0, 0).$$

In cylindrical coordinates, the vector operator  $\nabla^2$  applied to an arbitrary field  $(v_r, v_\theta, v_z)$  is

$$\begin{aligned} \nabla^2(v_r, v_\theta, v_z) = & \left[ -\frac{v_r}{r^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial r^2}, \right. \\ & -\frac{v_\theta}{r^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} + \frac{\partial^2 v_\theta}{\partial r^2}, \\ & \left. \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial r^2} \right]. \quad (6.4) \end{aligned}$$

Using

$$W_\mu = W_\mu(r) \exp(i\omega t + ikz + il\theta) \quad (6.5)$$

as an ansatz for the wave function, equation (6.3) becomes

$$\begin{aligned} \frac{d^2 W_0}{dr^2} + \frac{1}{r} \frac{dW_0}{dr} + (\omega + A^0)^2 W_0 - (m_W^2 + k^2 + \frac{l^2}{r^2}) W_0 \\ + \frac{2i}{r} \frac{\partial A^0}{\partial r} W_r = 0 \quad (6.6) \\ \frac{d^2 W_r}{dr^2} + \frac{1}{r} \frac{dW_r}{dr} + (\omega + A^0)^2 W_r - (m_W^2 + k^2 + \frac{l^2}{r^2}) W_r - \frac{2il}{r^2} W_\theta \\ - \frac{2i}{r} \frac{\partial A^0}{\partial r} W_0 = 0 \end{aligned} \quad (6.7)$$

$$\begin{aligned} \frac{d^2 W_\theta}{dr^2} + \frac{1}{r} \frac{dW_\theta}{dr} + (\omega + A^0)^2 W_\theta - (m_W^2 + k^2 + \frac{l^2 + 1}{r^2}) W_\theta \\ + \frac{2il}{r^2} W_r = 0 \quad (6.8) \end{aligned}$$

$$\frac{d^2 W_z}{dr^2} + \frac{1}{r} \frac{dW_z}{dr} + (\omega + A^0)W_z + (m_W^2 + k^2 + \frac{l^2}{r^2})W_z = 0. \quad (6.9)$$

It may immediately be seen that the  $z$ -component of the wave equation is in fact identical to the Klein-Gordon equation, (4.8). It follows, therefore, that  $W$  bosons are susceptible to the same sort of instability that pions may cause, though to a lesser extent due to the greater mass of the  $W$ . However, other forms of instability might arise from the other components of the wave equation.

It is possible to eliminate  $W_\theta$  from equation (6.7) by use of  $D_\mu W^\mu = 0$ .

Then, as long as  $l \neq 0$ ,

$$\frac{lW_\theta}{r} = \frac{i}{r} \frac{d}{dr}(rW_r) + (\omega - A^0)W_0 - kW_z. \quad (6.10)$$

It is also convenient to redefine the phase of  $W_0$ , so that  $W_0$  is replaced by  $-iW_0$ . Then equations (6.6) and (6.7) become

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + (\omega + A^0)^2 - k^2 - m_W^2 - \frac{l^2}{r^2} \right] W_0 - 2 \frac{dA^0}{dr} W_r = 0 \quad (6.11)$$

$$\begin{aligned} \left[ \frac{d^2}{dr^2} + \frac{3}{r} \frac{d}{dr} + (\omega + A^0)^2 - k^2 - m_W^2 - \frac{l^2 - 1}{r^2} \right] W_r \\ + \frac{2k}{r} W_z - \left[ \frac{2\omega}{r} - \frac{2}{r} A^0 + 2 \frac{dA^0}{dr} \right] W_0 = 0. \end{aligned} \quad (6.12)$$

In order to obtain a system amenable to analysis, it is necessary to assume that either  $k = 0$  or  $W_z = 0$ , to eliminate the coupling in (6.12). From considerations in Chapter 4, it is clear from the latter is not a good assumption, as the behaviour of the Klein-Gordon equation is not trivial in this case, but a restriction to  $k = 0$  may not be seen as too limiting in the light of the simplification it produces.

Then letting

$$W_0 = r^{-1/2}\psi_0$$

and

$$W_r = r^{-3/2}\psi_r,$$

we arrive at the equations

$$\left[ \frac{d^2}{dr^2} + (\omega + A^0)^2 - m_W^2 + \frac{\frac{1}{4} - l^2}{r^2} \right] \psi_0 - \frac{2}{r} \frac{dA^0}{dr} \psi_r = 0 \quad (6.13)$$

$$\left[ \frac{d^2}{dr^2} + (\omega + A^0)^2 - m_W^2 + \frac{\frac{1}{4} - l^2}{r^2} \right] \psi_r + 2(\omega + A^0 + r \frac{dA^0}{dr}) \psi_0 = 0 \quad (6.14)$$

which might be used to analyse possible instabilities caused by W bosons.

## 6.3 Conclusions

Unfortunately, further analysis of these equations proves very difficult, and requires more sophisticated techniques than those used in chapters 4 and 5. Because equations (6.13) and (6.14) are second order, it is necessary to match the derivatives of the wave functions as well as the wave functions themselves. This means, however, that there are four quantities which must be smoothly matched by variation of one eigenvalue parameter,  $\omega$ . The shooting and matching method discussed in earlier chapters is inadequate for this task. In the case studied by ANO, a relatively simple eigenvalue problem arose fortuitously, but this does not happen here.

It may be concluded that a vacuum instability does arise in the case of W bosons, similar to the charged condensate of pions, but less important because of the larger mass of the W. Further analysis is required to determine what, if any, other instability effects may be caused by W bosons.

## Chapter 7

### Conclusions

The results obtained in this thesis show that superconducting cosmic strings are certainly very complicated objects. They may be stabilized by their own supercurrents and charges, they emit gravitational and electromagnetic radiation, they are surrounded by clouds of charged bosonic particles, they collide with each other and, in short, seem to interact with each other and with other matter in virtually every conceivable way. Forming a consistent, coherent cosmological model that involves superconducting cosmic strings is thus a very difficult undertaking but, while this work has highlighted the magnitude of that task, it has also helped establish the issues that need to be addressed in obtaining such a picture. Several pointers to avenues of

future approach are apparent.

The interaction between a superconducting cosmic string and the surrounding plasma merits further study, as the shock-wave that a relativistic string produces must cause some damping of the string's motion and there may be some effect of the charged particles in the plasma on the rate of tunnelling of charged particles from the string.

As mentioned earlier, electromagnetic self-interactions should be studied. Plasma interactions would be important in this context, as the electromagnetic properties of the plasma would participate in determining the behaviour of electromagnetic fields in the vicinity of the string. It is also important to know what effects electromagnetism has on kinks and cusps.

Gravitational radiation from superconducting cosmic strings should be studied, because stabilized loops would radiate much less than free ordinary cosmic strings due to their more sedate motion. This would reduce the stochastic background radiation from cosmic strings, and resolve the looming conflict with pulsar timing observations [56,21]. Given some typical superconducting cosmic string trajectories, the method of reference [38] could be applied to obtain this information. The determination of what 'typical' trajectories are, however, is not a trivial task.

Extension of the work on spin- $\frac{1}{2}$  vacuum instabilities would give a better idea of its significance. Calculation of the density of higher energy states would provide an improved understanding of the relative importance of the fermionic vacuum instability and the pion condensate. The interaction between the two types of instability via pion decay also requires exploration.

As Ambjorn, Nielsen and Olesen have pointed out [47], a condensate of particles in the vicinity of a superconducting cosmic string must have some effect on the string itself, and may tend to reduce the current or charge on the string. An investigation of this possibility would be worthwhile. The treatment in the present work and in reference [47] ignores this possibility by regarding the field of the string as a fixed background field. Backreaction of the condensate on the string would be modelled by allowing the field of the string to have dynamical behaviour.

While it has been established that a charged condensate of W bosons can arise due to the z-component of the wave equation, the possibility of other types of instability due to the other components should be studied. The instability may be much more complicated than in the Klein-Gordon case, and might give rise to interesting physical phenomena. Some technique other than the shooting and matching method used in this work would be



needed, however, possibly relaxation.

At the moment, there is no over-all picture of the evolution of a universe dominated by superconducting cosmic strings. In the case of ordinary cosmic strings, numerical simulations of string networks, with strings colliding and intercommuting but not interacting significantly in any other way, have been used to produce such a picture [17,19,20,22]. Because of the complexity of superconducting cosmic string interactions, it is unlikely that such an approach would prove fruitful in this case, at least not until there is a better understanding of the underlying phenomena. The complexity of superconducting cosmic string behaviour is such that a thermodynamic description could be more instructive than a detailed attempt to model the dynamics of all the relevant phenomena.

# Appendix A

## Notation and conventions

Natural units, with  $c = \hbar = 1$  are used throughout.

The metric signature chosen is always  $(-1,1,1,1)$  for ordinary space-time and  $(-1,1)$  on a string world-sheet.

Greek indices run over the four dimensions of space-time,  $0, 1, 2, 3$ . The use of Roman indices varies with context, and is specified in each of the relevant chapters. In discussions involving string worldsheets, Roman indices are used for the two dimensions of the string worldsheet,  $0, 1$ , while in other chapters they are used for the three space-like coordinates of space-time,  $1, 2, 3$ . The symbols  $\tau$  and  $\sigma$  are also used to denote worldsheet coordinates.

Three-vectors are written in bold-face type, for example  $\mathbf{x}$ . Four-vectors, when denoted explicitly, are written

$$x^\mu = (x^0, \mathbf{x}).$$

The gradient operator on the worldsheet,  $\frac{\partial}{\partial \xi^i}$  is often abbreviated to  $\partial_i$ . A similar abbreviation is used for the spacetime gradient operator, where  $\partial_\mu = \frac{\partial}{\partial x^\mu}$ .

The Einstein summation convention is used for tensor expressions with matched upper and lower indices. Thus,

$$F^{\mu\nu} x_\nu = \sum_{\nu=0}^3 F^{\mu\nu} x_\nu$$

and

$$h^{ij} \partial_i Z^\mu = \sum_{i=0}^1 h^{ij} \partial_i Z^\mu.$$

The tension, or mass per unit length, of a string is denoted  $\mu$ , in accordance with common usage. In cases where four-vector notation is prevalent, and there is a possibility of confusion with the tensor index, as in  $x^\mu$ , for example, the string tension is instead denoted  $T$ . The gravitational constant is written  $G$ .

In discussions that deal with charged particles, the charge of individual particles is denoted by  $q$  and the mass of the particles by  $m$ .

The Kronecker delta,  $\delta_{ij}$ , is defined by  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  otherwise. The alternating tensor,  $\epsilon_{ijk}$ , is defined by  $\epsilon_{ijk} = 1$  when  $ijk$  is a symmetric permutation of 123,  $\epsilon_{ijk} = -1$  for anti-symmetric permutations, and  $\epsilon_{ijk} = 0$  if  $ijk$  is not a permutation of 123. A two-dimensional version,  $\epsilon^{ij}$ , is used on the string worldsheet, and is defined analogously, with  $ij$  a permutation of 01.

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